Flow electrification in rectangular channels - Comparison of different theoretical models

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***Abstract*— This paper deals with flow electrification phenomenon in channels of rectangular cross section. Different theoretical models are described and compared. For all the models it is assumed that the diffuse layer is fully developed. The space charge density conveyed by the flow is computed. First, two cases are examined in the case of weak space charge density, the exact rectangular channel solution is compared with the approximate solution of two parallel planes. Then, without any assumption on the level of the space charge density profile, and in the case of two parallel planes assumption, the charge conveyed is computed and compared to the solution obtained for a weak space charge density commonly assumed. Finally, the validity of each model is discussed.**

I. Introduction

The phenomenon of flow electrification has been investigated for more than fifty years now [1-5]. Thus, the influence of different parameters is now well understood. This is the case of the flow regime influence (specially the jump from laminar to turbulent flow), the influence of the Reynolds number and of the radius in the case of a pipe, the influence of roughness [1,6]. The physicochemical process at the origin of the phenomenon has been widely investigated as well [7,14] but unfortunately it remains badly understood and is still under investigation. Moreover, very few investigations have been made on the influence of the shape of the channel [15] and often an approximate solution available only for parallel plates and for the case of weak space charge density is employed. But, with the increasing of micro fluidic applications in various fields, it seems to be important to see the deviations between this approximate solution with the exact one or at least with a more realistic one. This is the purpose of this paper.

II. Important parameters

Apart from the hydrodynamic parameters of the flow and the geometrical parameters of the tube or channel, two important parameters control the flow electrification phenomenon. One is the Debye length: $\delta_0 = \sqrt{\frac{\varepsilon D_0}{\sigma}}$, where ε is the permittivity of the liquid, D_0 is the mean diffusion coefficient of the ions or dissociate impurities and σ the liquid electrical conductivity, this parameter determines the non dimensional diameter of the duct. The other parameter relates the intensity of ions exchange at the interface solid/liquid, we call it the space charge density on the wall: ρ_w. This parameter cannot be directly measured, it is generally obtained from experimental measurements of flow electrification. Indeed, we measure the space charge density transported for a laminar flow, for which the characteristics of the flow are well known, and in the case of a totally developed diffuse layer [5].

Then, depending on the model chosen for the velocity profile and the space charge density profile, from the value of the charge transported obtained experimentally we can find different values for the space charge density on the interface ρ_w.

III. Two parallel planes model

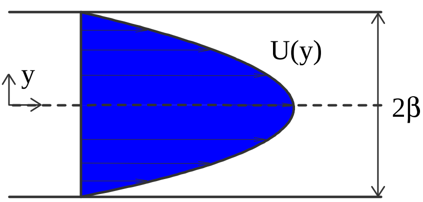

Fig. 1 Velocity profile

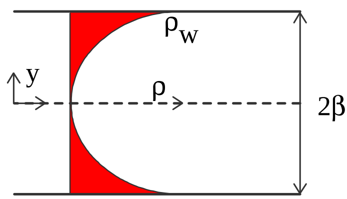

Fig2. Space charge density profile

The velocity is given by:

$$U(y) = \frac{3}{2} U_m \left(1 - \frac{y^2}{\beta^2} \right) \tag{1}$$

U_m being the mean velocity. With the assumption of weak space charge density, the space charge density profile is given by:

$$\rho(y) = \rho_w \frac{\cosh(y/\delta_0)}{\cosh(\beta/\delta_0)} \tag{2}$$

The charge transported by the flow, also sometimes called streaming current and the flow rate are (per unit width):

$$\int_{-\beta}^{+\beta} \rho(y) U(y) dy \qquad \int_{-\beta}^{+\beta} U(y) dy \qquad (3),(4)$$

Finally, the space charge density transported is:

$$Q = \frac{\int_{-\beta}^{+\beta} \rho(y) U(y) dy}{\int_{-\beta}^{+\beta} U(y) dy} \tag{5}$$

After some calculus we find:

$$Q = \frac{3\rho_w}{(\beta / \delta_0)^2} \left[1 - \frac{\tanh(\beta / \delta_0)}{(\beta / \delta_0)} \right] \tag{6}$$

Thus, knowing from experiment the space charge density transported Q_{exp} we can obtain the value of ρ_w. We call it ρpp because it is the case of parallel plane:

$$\rho_{wpp} = \frac{Q_{exp} (\beta / \delta_0)^2}{3\left[1 - \frac{\tanh(\beta / \delta_0)}{(\beta / \delta_0)} \right]} \tag{7}$$

The relation given in equation (7) is the most commonly used because of its simplicity but it is also based on two important assumptions rarely satisfied. The first one is the hypothesis of two parallel planes instead of a rectangular channel, this hypothesis could be realistic if and only if one size of the rectangular channel section is much greater than the other one. The second one is the hypothesis of weak space charge density, this is the case if the non dimensional space charge density in the whole section is much smaller than 1. Unfortunately we have often found in many experiments that it is greater or at least in the same order of magnitude than 1.

Following, again with the assumption of weak space charge density, we are going to compare this previous model with the exact solution in a rectangular channel.

IV. Charge transported in a rectangular channel

A. Laminar flow in a rectangular channel

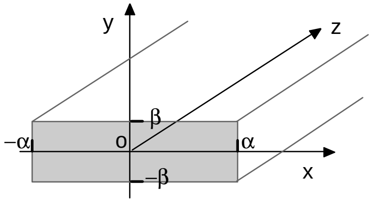

Fig.3Rectangular channel

We consider a rectangular channel with an axis in the z direction. The size is 2α in the x direction and 2β in the y direction. The Navier-Stokes equations for a laminar unidirectional flow of a non compressible fluid inside a tube of any kind of cross section can be written:

$$\Delta U = \frac{1}{\mu}\frac{dP}{dz} \tag{8}$$

and

$$\frac{\partial U}{\partial z} = 0 \tag{9}$$

U being the velocity in z direction, μ the dynamic viscosity and P the pressure.

On the limit of the domain (wall channel) we have Dirichlet conditions and the solution can be obtained with Green functions:

$$U = -\frac{1}{\mu}\frac{dP}{dz}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} 16\frac{\sin\left(\frac{(2k+1)\pi(x+\alpha)}{2\alpha}\right)\sin\left(\frac{(2l+1)\pi(y+\beta)}{2\beta}\right)}{\left[(2k+1)(2l+1)\pi^2\right]\left[\frac{(2k+1)^2\pi^2}{4\alpha^2}+\frac{(2l+1)^2\pi^2}{4\beta^2}\right]} \tag{10}$$

This solution is available inside the domain: $x \in \left]-\alpha,+\alpha\right[$ and $y \in \left]-\beta,+\beta\right[$, but not on the wall ($x = \pm\alpha$ or $y = \pm\beta$) for which U=0.

The mean velocity is:

$$U_{moy} = -\frac{1}{\mu}\frac{dP}{dz}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{64}{\left[(2k+1)^2(2l+1)^2\pi^4\right]\left[\frac{(2k+1)^2\pi^2}{4\alpha^2}+\frac{(2l+1)^2\pi^2}{4\beta^2}\right]} \tag{11}$$

The maximum velocity is for x=0 and y=0:

$$U_{max} = -\frac{1}{\mu}\frac{dP}{dz}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{16(-1)^k(-1)^l}{\left[(2k+1)(2l+1)\pi^2\right]\left[\frac{(2k+1)^2\pi^2}{4\alpha^2}+\frac{(2l+1)^2\pi^2}{4\beta^2}\right]} \tag{12}$$

We can see in Fig.4 the velocity profile in the case of a channel of square cross section ($\alpha = \beta$).

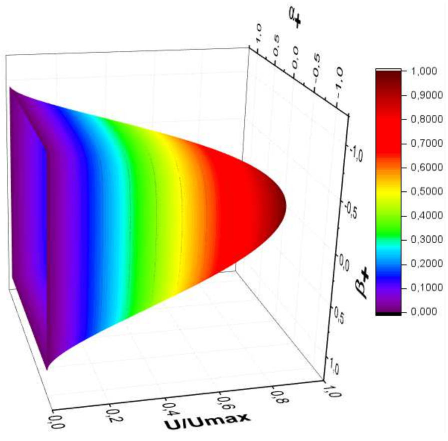

Fig.4. Velocity profile in a square cross section channel

B. Space charge density in a rectangular channel

With the assumption of weak space charge density we have:

$$\overrightarrow{\text{grad}}_{+}\rho_{+} + \gamma^{2}\,\overrightarrow{\text{grad}}_{+}\varphi_{+} = 0 \tag{13}$$

$$\Delta_{+}\varphi_{+} = -\rho_{+} \tag{14}$$

ρ_{+} being the non dimensional space charge density, φ_{+} the non dimensional potential and γ a coefficient taking into account the diffusion coefficient of ions, in practice it is always close to 1. The previous system can be written as follows:

$$\overrightarrow{\text{grad}}_{+}\left[-\Delta_{+}\varphi_{+} + \gamma^{2}\varphi_{+}\right] = 0 \tag{15}$$

Which is equivalent to:

$$\left[-\Delta_{+}\varphi_{+} + \gamma^{2}\varphi_{+}\right] = \text{cste} \tag{16}$$

We will take cste equal to the non dimensional space charge density on the wall of the channel:

$$\left[-\Delta_{+}\varphi_{+} + \gamma^{2}\varphi_{+}\right] = \rho_{w+} \tag{17}$$

This implies $\varphi = 0$ on the wall of the channel

The solution of this equation is, again, obtained with the help of Green function.

$$\rho_{+}(x_{+}, y_{+}) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} 16\,\rho_{w+}\,\frac{\left[\dfrac{(2k+1)^{2}\pi^{2}}{4\alpha_{+}^{2}} + \dfrac{(2l+1)^{2}\pi^{2}}{4\beta_{+}^{2}}\right]\sin\left(\dfrac{(2k+1)\pi(x_{+}+\alpha_{+})}{2\alpha_{+}}\right)\sin\left(\dfrac{(2l+1)\pi(y_{+}+\beta_{+})}{2\beta_{+}}\right)}{\left[(2k+1)(2l+1)\pi^{2}\right]\left[\dfrac{(2k+1)^{2}\pi^{2}}{4\alpha_{+}^{2}} + \dfrac{(2l+1)^{2}\pi^{2}}{4\beta_{+}^{2}} + \gamma^{2}\right]} \tag{18}$$

This solution is available on the open intervals:

$$x_{+} \in \left]-\alpha_{+}, +\alpha_{+}\right[\quad \text{and} \quad y_{+} \in \left]-\beta_{+}, +\beta_{+}\right[$$

On the wall, i.e. $x_{+} = \pm\alpha_{+}$ or $y_{+} = \pm\beta_{+}$, the solution is: $\rho_{+} = \rho_{w+}$

In dimensional form we have:

$$\rho(x, y) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} 16\rho_{w}\,\frac{\left[\dfrac{(2k+1)^{2}\pi^{2}}{4\alpha^{2}} + \dfrac{(2l+1)^{2}\pi^{2}}{4\beta^{2}}\right]\sin\left(\dfrac{(2k+1)\pi(x+\alpha)}{2\alpha}\right)\sin\left(\dfrac{(2l+1)\pi(y+\beta)}{2\beta}\right)}{\left[(2k+1)(2l+1)\pi^{2}\right]\left[\dfrac{(2k+1)^{2}\pi^{2}}{4\alpha^{2}} + \dfrac{(2l+1)^{2}\pi^{2}}{4\beta^{2}} + \dfrac{\gamma^{2}}{\delta_{0}^{2}}\right]} \tag{19}$$

We can see in Fig.5 the space charge density profile in the case of a channel of square cross section ($\alpha = \beta$) and for $\gamma = 1$.

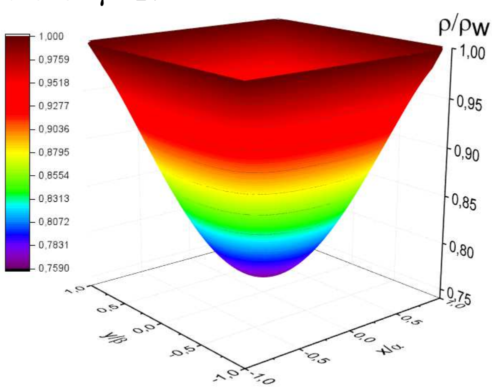

Fig.5. Space charge density profile in a square cross section channel

C. Space charge density transported by a laminar flow in a rectangular channel

We have to compute equation (5) in the case of a rectangular channel, then determine the space charge density on the wall from the experimental value of the space charge density transported. We call it ρ_{wrect} because it is the case of channel of rectangular cross section:

$$\rho_{wrect} = Q_{exp} \frac{\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{\left[(2k+1)^2(2l+1)^2\right]\left[\frac{(2k+1)^2\pi^2}{4\alpha^2}+\frac{(2l+1)^2\pi^2}{4\beta^2}\right]}}{\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{\left[(2k+1)^2(2l+1)^2\right]\left[\frac{(2k+1)^2\pi^2}{4\alpha^2}+\frac{(2l+1)^2\pi^2}{4\beta^2}+\frac{\gamma^2}{\delta_0^2}\right]}} \quad (20)$$

We can see in Fig. 6 the ratio of the value obtained with the assumption of parallel planes on the value obtained with the rectangular cross section channel solution. This ratio is computed for 10 different values of α/β varying from 1 to 10 and 100 values of β/δ_0 varying between 1 to 100. We can see that when α/β is large the results obtained with the two models are quite close, this is normal because, for such cases, the cross section is more like parallel planes. On the other hand, the greater β/δ_0, the greater is the difference between the predictions given by the two models.

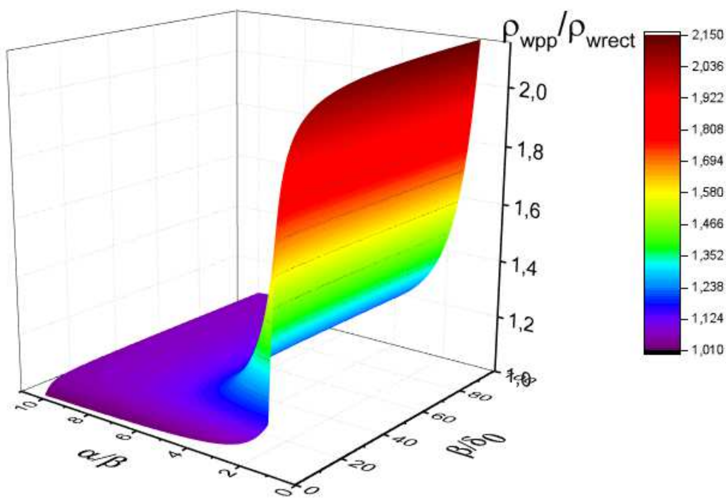

Fig.6. Comparison between the paralell plane and rectangul model

V. TWO PARALLEL PLANES MODEL WITHOUT WEAK SPACE CHARGE ASSUMPTION

We are going to compute the space charge density transported by laminar flow through parallel planes for which β/δ_0 is large enough in order to consider that the expression of the space charge density on each plane is the same that the one existing on a single plane (in a semi infinite domain). The space charge density on a single plane without any assumption on the magnitude of the space charge density has already been computed [16]. For this computation we take the origin on the plane ($y = \beta$ corresponds to the median plane).

$$\rho = 2\,\rho_{ref}\,\frac{\cosh(y_+ + C)}{\sinh^2(y_+ + C)} \qquad \text{with: } C = \mathrm{ArgCosh}\left[\frac{1}{\rho_{w+}} + \sqrt{1 + \frac{1}{\rho_{w+}}}\right] \tag{21}$$

ρ_{ref} being the space charge density of reference ($\rho_+ = \rho / \rho_{ref}$)

With y=0 on the wall plane the velocity is given by:

$$U = \frac{3}{2} U_m \left(\frac{2y}{\beta} - \frac{y^2}{\beta^2}\right) \tag{22}$$

U_m being the mean velocity. Finally, the non dimensional space charge density transported by the flow is:

$$Q_+ = \frac{3}{\beta_+}\int_0^{\beta_+} \frac{\cosh(y_+ + C)}{\sinh^2(y_+ + C)} \left(\frac{2y_+}{\beta_+} - \frac{y_+^2}{\beta_+^2}\right) dy_+ \tag{23}$$

Integration gives:

$$\begin{aligned} Q_+ = {}& \frac{6}{\beta_+}\left[\frac{-1}{\sinh(\beta_+ + C)} + \frac{1}{\beta_+}\left(\ln\left(\tanh\left(\frac{\beta_+ + C}{2}\right)\right) - \ln\left(\tanh\left(\frac{C}{2}\right)\right)\right)\right] + \\ & \frac{3}{\beta_+ \sinh(\beta_+ + C)} - \frac{6}{\beta_+^2}\ln\left(\tanh\left(\frac{\beta_+ + C}{2}\right)\right) + \\ & \frac{6}{\beta_+^3}\left[\ln\left(\tanh\left(\frac{\beta_+ + C}{2}\right)\right)\ln\left(1 + \tanh\left(\frac{\beta_+ + C}{2}\right)\right) + \mathrm{li2}\left(1 - \tanh\left(\frac{\beta_+ + C}{2}\right)\right) + \mathrm{li2}\left(-\tanh\left(\frac{\beta_+ + C}{2}\right)\right)\right] \end{aligned} \tag{24}$$

This value must be compared to the value obtained with the weak space charge assumption:

$$Q_+ = \frac{3\rho_{w+}}{(\beta_+)^2}\left[1 - \frac{\tanh(\beta_+)}{(\beta_+)}\right] \tag{25}$$

We can see in Fig. 7 this comparison

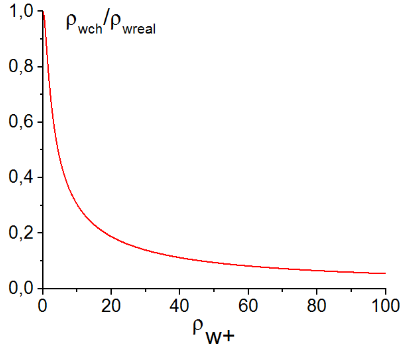

Fig.7. Comparison between the simple model and the reality

VI. Conclusion

This paper concerns flow electrification in rectangular channels. We made a comparison between the simple model, commonly used, making the assumption of weak space charge density and considering that the rectangular channel can be assumed identical to two parallel planes. Firstly, we compared, in the case of weak space charge density, the prediction with a rectangular model and that given for two parallel planes; we found a ratio between the predictions up to 2.2. Secondly, we compared, in the case where the hypothesis of parallel plane can be assumed ($\alpha > 10\,\beta$), and for $\beta \geq 10\,\delta_0$, the results predicted by the simple model with the real value. We found a ratio up to 20.

This means that we must be careful using the simple model.

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