Some Comments on the Charge Decay Paradox in Metals

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***Abstract*—Almost all textbooks written in the last 100 years that discuss charge decay in a good conductor such as a metal do so by combining the equation of continuity with Ohm's law and Gauss's law and solving the modified continuity equation to obtain an exponential decay of the charge density with time. In this analysis the power of the exponent is the ratio of time to a characteristic time. For a good conductor this characteristic time, referred to as the material's electrical relaxation time constant, is very small – typically less than 10^{-18} second (copper: approximately 1.5 x 10^{-19} s). If free charges are uniformly placed in a good conductor, coulomb repulsion occurs, and the charges are repelled to the outside of the conductor. As a result the charges disappear from the volume of the conductor within a few time constants. It is argued, for example, that if the added charges in a copper conductor move with an average drift velocity u then in a relaxation time they will move a distance of approximately 10^{-19} u. For a copper wire of 2 mm diameter, a charge located very near the center of the wire would need to move with a drift velocity u of approximately 10^{16} m/s to reach the surface of the wire in one time constant. The speed of light is only 10^{8} m/s; so, based on the above logic, charges would have to travel in excess of the speed of light. This paradox has caused much discussion in the scientific community over the ensuing years. A new analysis of the situation is presented in this paper which eliminates the paradox.**

I. Introduction

Ohm's law is an important and very useful law that is applied to many problems, not just in electronics, but also in electrostatics. In electronics Ohm's law is usually expressed in the circuit form $V = IR$ which stated when a voltage V is placed across a material of a specified shape a current I passes through the material, and the ratio of V/I is a constant – this constant being known as the resistance R of the specific shaped material. What is so important about Ohm's law is that this resistance is equal to a geometry factor – defined by the shape of the material – divided by another constant known as the conductivity σ_m of the material. It was James Clerk Maxwell [1] who pointed out Ohm's law would have little scientific value if a specific property of the material (i.e., its conductivity) could not be defined as independent of the strength of the current flowing through it and the electric potential at which the conductor is maintained. Maxwell noted that the material's conductivity "depends entirely on the nature of the material of which the conductor is com-

posed." Thus, the constant σ_m is a true property of the material.

When it is desired to understand what is going on everywhere inside a material, it is convenient to describe what happens in a little volume element within the material and then to shrink the volume element to a point. Then, by summing up or integrating the information found at all these points an understanding of bulk measurable quantities can be obtained. When examining an incremental volume element, the field form of Ohm's law becomes more useful than the circuit form. The field form of Ohm's law at any point in a metal is written as

$$\mathbf{J} = \sigma_m \mathbf{E} \tag{1}$$

where σ_m is the electrical conductivity of the metal, **J** is the charge transport or current per unit area flowing through the point and **E** is the electric field at that point. The field form of Ohm's law can be used to derive the circuit form of Ohm's law. However, the field form can also be used to investigate what happens inside an ohmic material.

Ohm's law is usually assumed to be the appropriate law for charge transport in a metal conductor. One consequence of the field form of Ohm's law is that in total thermodynamic equilibrium the electric field everywhere inside the conductor must be zero. This can be seen in (1), since if **E** were not zero, there would be a charge transport **J;** and this current density could be used to operate a perpetual motion machine, which is thermodynamically impossible.

As a result of the electric field being zero when the conductor is in thermodynamic equilibrium, it is usually argued that, if some charges, all of the same polarity, are uniformly placed inside this isolated conductor, Coulomb's law of repulsion will force the charges to move away from each other causing the charges to eventually reach the surface of the conductor.

If a uniform charge density ρ is placed inside an isolated metal conductor in thermodynamic equilibrium, the question is: how long does it take for these charges to get to the surface? The answer, found in most textbooks, is obtained by substituting Ohm's law into the equation of continuity

$$-\partial \rho / \partial t = \nabla \cdot \mathbf{J} \tag{2}$$

and then applying Gauss's law

$$\nabla \cdot \varepsilon_m \mathbf{E} = \rho \tag{3}$$

– under the assumption of an isotropic and homogeneous permittivity where the left hand side of (3) becomes $\nabla \cdot \varepsilon_m \mathbf{E} = \varepsilon_m \nabla \cdot \mathbf{E}$ – to obtain a modified continuity equation

$$-\partial \rho / \partial t = (\sigma_m / \varepsilon_m) \rho \quad . \tag{4}$$

This differential equation has the solution

$$\rho = \rho_0 e^{-t/\tau_m} \tag{5}$$

where ρ is the charge density in the metal conductor at any time t, while ρ_0 is the charge density at the initial observation time $t_0 = 0$, and

$$\tau_m = \varepsilon_m / \sigma_m \tag{6}$$

is referred to as the material's electrical relaxation time constant which is given by the ratio of two constants: the material's permittivity ε_m and its electrical conductivity σ_m.

Copper is considered a very good conductor. Copper has an electrical conductivity of 5.7 x 10^7 S/m (siemens per meter) and a dielectric constant ε_{rm} near unity; so, its permittivity $\varepsilon_m = \varepsilon_{rm}\varepsilon_0$ is close to the permittivity of free space which is ε_0 = 8.85 x 10^{-12} F/m (farads per meter). As a result, the electrical relaxation time for copper is very short $\tau_m \approx$ 1.5 x 10^{-19} s. From (5) it can be seen that in a time $t = \tau_m$ the charge density will be only about 37% of its originally observed value, in $5\tau_m$ less than 1%, and in $10\tau_m$ less than 0.005% of its value. Because the electric field must be zero in total thermodynamic equilibrium and because like charges repel, it is reasoned that the charges, due to coulomb repulsion, are forced to go to the surface of the conductor.

A. The Ohm's law (or charge decay) paradox

The Ohm's law paradox of classical physics is essentially an argument that says when the classical physics equations – Gauss's law and the equation of continuity – are used, along with the equation known as Ohm's law, to determine the charge decay in a metal, the result implies the charges would have to travel faster than the speed of light in order to reach the surface of a conductor of any reasonable size. This author is not sure when the paradox first appeared in the scientific literature, but the interested reader will find an excellent discussion of the problem in a series of Letters in the Proceedings of the IEEE during the 1970s (see below). Just the key points of the paradox are mentioned here.

1) The paradox

In 1972 Mott et al. [2] argued that intuitively the charge decay equation (5) must not be correct since this charge decay equation is independent of the size of the conductor.

They also reasoned that for a good conductor τ_m is on the order of 10^{-19} s; so, if most of the charges appeared on the surface in say $10\tau_m \approx 10^{-18}$ s, the charges would travel with speed u a distance of $L = \underline{ut} = 10^{-18}\, u$ meters. If the charges were not allowed to move faster than the speed of light, which is 3 x 10^8 m/s, then in $10\tau_m$ the charges would only move on the order of 10^{-10} m. This is roughly on the order of atomic hydrogen's diameter.

From the above it was concluded that for a metal of macroscopic dimensions it would be impossible for a charge to be removed from well within the interior of the metal unless a charge traveled faster than the speed of light, which – based on relativity – is a physical impossibility. Since the decay was based on Ohm's law, this contradiction with relativity became known as the Ohm's law paradox. It is also called the charge decay paradox.

2) Some discussion of the paradox

The paradox listed above led Mott et al. [2] to suggest that the charge decay analysis that gives the exponential decay as the ratio of ε_m/σ_m must be incorrect. Mott et al. went on to suggest that the electrical conductivity was not constant when charges are added to a metal conductor and that the correct analysis would require keeping the mobility constant and allowing the conductivity to vary in time.

Mott et al. [2] included a diffusion term in their charge transport equation, thus negating the use of Ohm's law whenever excess charge is placed in a conductor. They did not solve their equation in general, but did present a solution when only diffusion occurred.

Over the next few years (1973 through 1976) a flurry of papers followed [3], [4], [5], [6], [7] and [8]. Each paper and the replies each paper generated added to the discussion in what this author can only described as a vigorous and healthy interaction on a most important and fundamental topic. With one exception [5], all this discussion appeared in the *Proceedings of the IEEE* in its Letters section; but the Letters section of this publication was soon thereafter abandoned – which, unfortunately, eliminated the main forum for further discussion. In 1976 Sarma [7] and Torrens [8] presented the last of the informational papers; but in the end the question still remained as to why the charge decay paradox occurs in a metal (i.e., a good) conductor (see [8] p. 280 last paragraph).

A few years later Chanian [9] brought up the controversy again. He listed several introductory textbooks written in the last half of the 20th century which combined, as discussed above, the equation of continuity, Ohm's law and Gauss's law to arrive at (5). He also argued these textbooks are all in error. Chanian raised several points, but the major argument was that for such a short relaxation time Ohm's law is not valid because Ohm's law is a balance between the momentum gained from the electric field and the momentum lost between collisions. For copper the collision time is about 2 x 10^{-14} s; so a time at least that long is required for this balance to occur.

Chanian attempted to put the question to rest by arguing that charge decay was actually based on the slowest time of a multi-step process. First he noted Saslow and Wilkinson's [10] suggestion that free-electron collisions caused these charges to create damped plasma oscillations, and the charges were removed in approximately twice the collision time (about 4 x 10^{-14} s for copper). Chanian then argued that there were other longer-lasting, transient phenomena in the conductor. He examined these transients (after $\rho = 0$), and found the electric and magnetic fields in the bulk conductor decayed based on diffusion equations. He then looked at the relaxation time for the surging surface currents at the conductor's surface and argued that the true relaxation time is the longer of the three times (the time to move charges to the surface, the time to diffuse the fields out of the volume and the time for the surface currents to relax). While such calculations are interesting, they still maintain the same paradox. Namely, in Chanian's method the charges leave the volume of the conductor in a short time (twice the collision time) so that it would still appear that the charges at the center of the conductor must move faster than the speed of light to reach the surface in a few relaxation times and the paradox remained.

Almost a decade ago, Seaver [11] developed the charge flux equation in a general material based on a contiguous collision averaging method. Under the conditions that the material is linear, isotropic, homogeneous, and non-flowing (like a solid), this equation reduced to an equation that looks just like Ohm's Law. Later Seaver [12] applied the charge flux equation to a solid material that had uniform charge placed within it, and he developed a single charge decay equation valid for any solid whether it be a conductor, a semiconductor or an insulator. However, his equation also contained the exponential decay term and is therefore subject to the same paradox.

It is the purpose of this paper to reexamine the paradox. It will be shown that a "real" perturbation charge in a metal can only be a free-electron, and it does not have to travel to the surface in order for charge decay to occur. Furthermore, it will be shown that a relaxation time on the order of 10^{-18} s is reasonable for good conductors and that this time does not require a perturbing charge to travel at a speed anywhere near the speed of light.

Most of the discussion in this paper is very general, but at periodic intervals a 2 mm diameter copper wire of very long extent will be used as an example for calculation purposes. This example is chosen in order to show the discussion in this paper is relevant to a typical conductor found in a common geometry.

II. Discussion

The fact that the classical Ohm's law paradox has remained in the technical literature for so many years suggests that understanding how to unravel the paradox is not obvious, even if the answer – once seen – may seem trivial. Therefore, it seems appropriate to step back and take a look at all the basic physics involved in Ohm's law and determine if some things might have been overlooked in the discussion of charge relaxation. As will be shown in this paper (and marked by ►) there are several points that needed discussion in order to understand what really occurs during electrical charge relaxation in a metal.

Many calculations will be done in this paper, and some important constants used in these calculations are given in Table 1.

Table 1: Some useful fundamental constants

Name	Value
Avogadro's Number: A_N	6.022×10^{23} molecules/mol
Boltzmann's constant: k	1.38×10^{-23} J/K
Electron charge: $s_e q_0 = - q_0$	$- 1.6 \times 10^{-19}$ C
Electron mass: m_e	9.11×10^{-31} kg
Classical electron radius: r_e	2.82×10^{-15} m

In order to look at the basic physics in Ohm's law requires an understanding of charge flux. Once charge flux is understood, it becomes possible to understand when Ohm's law is valid and when it is not valid. Once Ohm's law is understood, questions can be asked and answered regarding what happens when charge is inserted into a conductor. For example, what type or types of charge and how much charge can be inserted in a conductor and still have the conductor be the original conductor? Once these questions have been answered the paradox question can be asked. Namely, can a charge placed deep within the interior of a good conductor take part in electrical relaxation within about 10^{-18} s and not travel faster than the speed of light? If this can happen, then the Ohm's law solution that gives charge decay as (5) is correct; otherwise, the paradox remains.

To obtain the basic understanding of Ohm's law, to ask and answer the relevant questions and to arrive at a clear and crisp understanding of charge decay which will clear up the Ohm's law paradox requires a trip through the several subsections of this paper.

A. Charge Flux

Ohm's law (1) is nothing more than a statement about charge flux. Ohm's law simply states: the charge flux or current density **J** is proportional to, and in the direction of, the

electric field **E** and the electrical conductivity σ_m is the constant of proportionality.

1) General charge flux equation

As already mentioned, Seaver [11] developed the charge flux equation for any charged species; say the ith species, in a general material (solid, liquid or gas). In the derivation he assumed any rate of generation R_{GENi} of the species was equal to the rate of recombination R_{RECi} of the species and that magnetic field effects could be ignored. He found the charge flux of this i^{th} species was

$$\mathbf{J_i} = \rho_i \mathbf{v_{d0}} + \sigma_i \mathbf{E} - D_i \nabla \rho_i - G_i \rho_i \nabla T \quad . \tag{7}$$

Here the current per unit area or charge flux $\mathbf{J}_i$ at any point in the material is dependent on the charge density ρ_i, the electrical conductivity σ_i, the electric field **E** and temperature T at that point. For liquids and gases the material might have a bulk material drift velocity $\mathbf{v}_{d0}$, but for a stationary metal like a copper wire – which is the example metal being considered here – this material drift is zero.

In (7) charge density ρ_i, conductivity σ_i, diffusion coefficient D_i and thermophoresis coefficient G_i for the ith species are given, respectively, by

$$\rho_i = s_i q_i n_i \quad , \tag{8}$$

$$\sigma_i = s_i^2 q_i n_i b_i \quad , \tag{9}$$

$$D_i = k T b_i / q_i \quad , \tag{10}$$

$$G_i = k b_i / q_i \quad , \tag{11}$$

where the mobility b_i of the ith species is given by

$$b_i = q_i \tau_{ci} / m_i \quad . \tag{12}$$

Here, q_i is the amount (but not the sign) of charge of the species of mass m_i which undergoes collisions with an average collision time τ_{ci}. This charge of the species is given by

$$q_i = Z_i q_0 \quad , \tag{13}$$

where q_0 = 1.6 x 10^{-19} C is the fundamental charge of a proton or positron and is a positive number, Z_i is the number of fundamental charges on the ith species and s_i is the sign of the charge; s_i = 1 if the charge is positive, and s_i = -1 if the charge is negative.

If more than one species of charge is present in the material, the total charge flux is given by the vector sum of the individual charge fluxes as

$$\mathbf{J} = \sum_i \mathbf{J}_i \quad . \tag{14}$$

When (14) is applied to (7) the total charge flux in the material is

$$\mathbf{J}=\rho\,\mathbf{v}_{\mathbf{d0}}+\sigma\,\mathbf{E}-\sum_i D_i\nabla\rho_i-\nabla T\sum_i G_i\rho_i \quad , \tag{15}$$

where the total charge density is

$$\rho=\sum_i \rho_i \quad , \tag{16}$$

and the total electrical conductivity is

$$\sigma=\sum_i \sigma_i \quad . \tag{17}$$

Clearly, (15) is not Ohm's law; but as Maxwell [1] noted, Ohm's law is only applicable to homogeneous conductors. Hence, if the material is a homogeneous solid conductor, there can be no charge gradients, $\nabla\rho_i = 0$, and no thermal gradient, $\nabla T = 0$, and no material flow, $\mathbf{v}_{\mathbf{d0}} = 0$, and for these (homogeneous conductor) restrictions (15) reduces to

$$\mathbf{J}=\sigma\,\mathbf{E} \quad . \tag{18}$$

Notice that (18) looks like Ohm's law (1) and (18) can be used as Ohm's Law if, and only if, it can be shown that σ in (18) is a constant; i.e., if, and only if: $\sigma = \sigma_m$.

Finally, for a homogeneous material only (8), (9), (12), (13) and (16) are needed to define the general conductivity (17) and once (17) is known the charge flux (18) of the homogeneous conductor can be computed.

2) Charge flux, Ohm's law and conductivity

In the previous section it was shown that the general charge flux equation (15) reduces to (18) for a homogeneous material. Now the goal is to see if the conductivity as given by (17) is a constant so that (18) can be used as Ohm's law for a metal conductor.

An atom can be considered as a group of electrons surrounding a positive ion core. In a metal each atom has one electron that is loosely bound to its ion core. This electron can move freely when acted on by a force and is called a free-electron. In a metal it is estimated that there is roughly one free-electron associated with each atom. As a result, in a metal there are two charge species, the free-electron species and the atom's ion core species that remains when free-electrons move from their atoms.

Although the ion cores can vibrate around their equilibrium positions, they are locked into the lattice structure that makes up the solid; and – when acted on by an external force – they cannot be dislocated or moved from their central positions. Thus, the mobility b_i of the ion core species is zero and by (9) the conductivity of the species is zero.

On the other hand, the free-electrons can be moved from their positions when acted on by an external force; so, they do have mobility and (9) – with the aide of (8), (12) and (13) – shows the conductivity of the free-electrons is finite and depends on the number density $n_i = n_e$ of free-electrons in the material.

Since, as noted above, the conductivity of the ion species is zero, from (17) these two conductivities add to give the total conductivity of the metal as

$$\sigma=s_e^2 q_0 n_e b_e \equiv \sigma_m \tag{19}$$

where the subscript e refers to the free-electron species; and the subscript e is also a reminder that the metal's conductivity is due entirely to the free-electrons in the metal.

A specific metal composition has atoms of a specific atom number density n_a and, if $n_a = n_e$, it also has that same specific free-electron number density n_e. From (19) it can be seen that the mobility b_e and the number density n_e of the free-electrons are what determine the value of the metal's conductivity (19); and, as a result, if the mobility and number density are constant then σ is a constant and can be written as σ_m.

So, in summary, for a metal (18) and (19) combine to give Ohm's law (1) under the assumptions that the mobility b_e and number density n_e are both constants. Stated another way, it can be concluded that:

►**i)** Ohm's law (1) is valid provided the free-electron mobility b_e and the free-electron number density n_e are both constant which makes a constant conductivity.

Is the mobility a constant? Measurements using the circuit form of Ohm's law always show that $\sigma = \sigma_m$ is a constant; so, clearly from the measurements one can conclude this to be true. However, later in this paper, it will be shown that, if the measurements were to have been made at a very much higher electric field (i.e., a very much higher potential difference across the metal), the mobility would no longer be constant. It will also be shown that just such a high field will occur if the initial perturbing charge density ρ_0 in (5) is too high. Furthermore, it will be shown that this restriction on the perturbing charge density ρ_0 can be calculated for any metal but is dependent on the size of the conductor.

B. Adding Perturbing Charge in a Metal

In the mind's-eye laboratory adding charge to a metal is just a simple thought experiment. But even in this virtual laboratory of unlimited resources some additions are not allowed.

1) Restrictions on the added charge

When a charge density ρ_0 is added into a good conductor such as a metal (5) indicates this charge density decays exponentially with time and with a time constant on the order of 10^{-18} s which some have intuitively argued is too quick for relaxation to occur.

If no thought is given to the addition of ρ_0, one might assume the added or perturbing charge density can be either positive or negative.

However, with some thought it can be realized that, for an isolated conductor, it is not possible to remove electrons from the conductor and use Ohm's law. The reason is, when some of the free-electrons are removed, this changes n_e, which, according to (19) changes σ_m, i.e., σ_m is no longer its original constant. As a result, the metal's composition is no longer its original composition and Ohm's law cannot be used. Hence, it is not possible to place positive charge in an isolated conductor by removing electrons.

Likewise, if atoms or atomic ions of any kind are inserted into the space of existing atoms in the conductor, Ohm's law is violated because the material composition has permanently changed which will also change σ_m. For example, if the perturbing atoms are of slightly different size, then, even if each has one free-electron, the charge <u>density</u> of the free-electrons will change and σ_m will change.

Some might suggest putting in positrons to give ρ_0 a positive sign; but, as soon as a positron encounters an electron, both the positron and electron would be annihilated, and the loss of the electron would change n_e which changes σ_m.

So the only thing that is left to do is to place more free-electrons into the metal. In this

situation the original composition of the species (i.e., the free-electrons and their ion core atoms) that made up the metal are still present in the proper amounts so that the metal still contains all the free-electrons that made σ_m a constant. However, it is not allowed to just stick free-electrons at some location and watch the decay with time because one of the requirements of an ohmic material is that it must be homogeneous, which requires $\nabla\rho_p = 0$. Hence, it can be concluded that:

►**ii)** The two restrictions on the added or perturbing charges are: 1) they must be free-electrons, and, 2) they must be added uniformly throughout the conductor.

Essentially, the above conclusion is based on the above discussion which found that it is impossible to add positive charge or atomic ions of any kind to an isolated ohmic conductor and have it be the same ohmic material it was before the addition.

2) Effect of the added charge on conductivity

When charge is uniformly added to any material (good or poor conductors), the charge flux equation must be reexamined. Uniform addition of charge means no gradients; so, (15) still reduces to (18). When charge is added, the procedure to determine the conductivity is the same as discussed previously, except now there is a third species present; i.e., the added or perturbing species. Without, for the moment, defining what that species is (although in the previous section it was shown for a metal it is free-electrons), (17) indicates that the perturbing species' conductivity must be included and (19) is replaced by

$$\sigma = \sigma_m + \sigma_p \tag{20}$$

where the subscript p refers to the species as the perturbing species. It is clear that the true charge flux equation for a uniformly placed perturbing species is (18) with the conductivity given by (20) and not (19). As a result, it can be concluded here that in general Ohm's Law is not the appropriate equation to use when perturbing charges are present.

However, if it can be shown that the perturbing conductivity σ_p is very much less than the material's conductivity σ_m, then an argument can be made that (20) is, for all practical purposes, a constant and Ohm's law can still be used. It will be shown later that, to insure the ohmic behavior of a metal conductor, the requirement of $\sigma_{p0} << \sigma_m$, must always exist.

3) Adding charge to a general material and to a metal

So what must be done to determine the charge decay in a general ohmic material due to a perturbing species? The answer is the equation of continuity (2) and Gauss's law (3) must be combined with the constitutive equation given by (18) and with the conductivity given by (20) to determine charge decay. This calculation has already been done for a general non-flowing material (solid or a stagnant high viscosity liquid or gas) composed of a single conducting species to which a second conducting species is added as an initial perturbation charge density ρ_{p0} and the result is [12]

$$\rho_p(t) = \frac{\rho_{p0} e^{(-t/\tau_m)}}{1 + \frac{\tau_m}{\tau_p}\left[1 - e^{(-t/\tau_m)}\right]} \tag{21}$$

where a perturbation time constant is defined as

$$\tau_p = \frac{\varepsilon_m}{(s_p \rho_{p0}) b_p} = \frac{\varepsilon_m}{s_p^2 q_p n_{p0} b_p} = \frac{\varepsilon_m}{\sigma_{p0}} \quad . \tag{22}$$

For the special case of a metal conductor where free-electrons are the only means of conduction (21) reduces to

$$\rho_p(t) = \frac{\rho_{p0} e^{(-t/\tau_m)}}{1 + \frac{n_{p0}}{n_e}[1 - e^{(-t/\tau_m)}]} \tag{23}$$

and, if $n_{p0} \ll n_e$, i.e., if the initial perturbing free-electron density is much less than the normal free-electron density of the material, then (23) reduces to (5).

So, where is the direction of this analysis heading? It will be shown later that for a metal to be defined as ohmic the assumption $n_{p0} \ll n_e$ will always be true because there is a limit to the magnitude of the perturbing charge density that can be added to a metal, above which further addition will alter the metal's conductivity by altering its free-electron mobility. In other words, if the initial perturbing charge density is too high, the conductivity is no longer a constant; and Ohm's law is no longer valid. On the other hand, if the initial perturbing charge density is below a certain amount (the amount depends on the size of the conductor), the conductivity is a constant, Ohm's law remains valid and charge decay defined by (23) reduces to the charge decay defined by (5). So, it may take some further discussion to get there, but one goal is to show $n_{p0} \ll n_e$ is always true for a metal, and that will be the defining argument in showing (5) is the correct charge decay equation for a metal. Then the Ohm's law paradox regarding the speed of light can be addressed without further questioning the appropriateness of either (1) or (5).

C. Free-electron Number and Charge Density

Table 2 lists useful information on copper. The upper part of Table 2 listed several of the properties of copper down through and including its measured electrical conductivity. From this information the remaining information in Table 2 will be calculated.

For a charge-neutral metal with a mass density ρ_D and a gram-molecular-weight M_A the atom number density is given by

$$n_a = A_N \rho_D / M_A \tag{24}$$

where A_N is Avogadro's number as given in Table 1. For the density and molecular-weight of copper as given in Table 2, copper's atom number density, using (24), is calculated to be 8.46 x 10^{28} atoms/m^3. Most metals have an atom number density within two orders of magnitude below this range.

If for any reason some of the free-electrons are not free, then the number density of the free-electrons will only be a fraction of the atom number density; namely

$$f_e = n_e / n_a \quad . \tag{25}$$

Table 2: Measured and calculated properties of copper

Name	Value	Method
Mass density: ρ_D	8930 kg/m^3	Periodic Table
Molecular weight: M_A	63.55 x 10^{-3} kg/mol	Periodic Table
Electrical conductivity: σ_m	5.7 x 10^7 S/m	Measured via. Ohm's law
Electrical relaxation time: $\tau_m \approx$	1.5 x 10^{-19} s	From (6)
Atom number density: n_a	8.46 x 10^{28} m^{-3}	From (24)
Free-electron number density: n_e	8.46 x 10^{28} m^{-3}	From (25) with $f_e = 1$
Ion core number density: n_{a+}	8.46 x 10^{28} m^{-3}	Charge neutrality $n_e = n_{a+}$
Electron mobility: b_e	4.24 x 10^{-3} m^2/V·s	From (19)
Electron collision time: τ_{ce}	2.4 x 10^{-14}s	From (12)
Electron thermal speed: v_{th_e}	1.06 x 10^5 m/s	From (42) at 293 K
Ohm's law max drift velocity: $\mathbf{v}_{d_e}$	1.15 x 10^4 m/s	From (53) at 293 K
Ohm's law copper max-field: E_{max}	2.7 MV/m	From (19), (42), (54)
Max free-electron movement: δ_r	1.78 x 10^{-15} m	From (65) $r = r_w = 1$ mm

However, in a metal it is usually assumed that there is one free electron per atom. So, with the assumption $f_e = 1$, (25) gives the free-electron number density for copper as 8.4 x 10^{28} charges/m^3. This means there are about 4390 charges/µm or that the charges are separated by about 2.3 x 10^{-4} µm = 0.23 nm = 2.3 Å from each other. In fact, the atoms in most metals are separated from each other by only a few angstroms.

From (8) the free-electron charge density is $s_e q_0 n_e = -q_0 n_e$ which for copper is -1.35 x 10^{10} C/m^3. This result does not mean there is a large charge in the wire. On the contrary charge-neutrality exists because each free-electron is associated with a metal ion in the wire. Charge neutrality will always require the positive ion core density n_{a+} to be equal to the free-electron number density when in total thermodynamic equilibrium. In other words, $n_e = n_{a+}$ when in total thermodynamic equilibrium. As a result, the summation as defined by (16) always makes for a neutral metal wire.

Now for a very important point. Even when thermodynamic equilibrium is abandoned, as for example when a current is made to flow, charge-neutrality remains in an ohmic conductor such as a metal wire. When a current is made to flow, a free-electron moves from one ion core to the next until eventually it moves out of the wire. However, since current-in equals current-out, for each free-electron that leaves the wire, a new free-electron enters (otherwise there would be a $\nabla\rho_e$ and the wire would no longer be ohmic). At any given instance a given free-electron that entered can be associated with one of the atomic ions. When the current is made to stop, there will still be one free-electron associ-

ated with each atomic ion – the sum of which makes up a neutral material. However, an important conclusion can be drawn from this little discussion. Namely, from this intuitive understanding of charge motion in a wire it can be concluded that:

► **iii)** When free-electrons are made to move in a metal, and after which thermodynamic equilibrium is established, charge neutrality of the atoms exists, but not necessarily with a specific atomic ion being associated with the same free-electron it had been associated with prior to the cause of free-electron motion.

The above conclusion will be very important in understanding how the electrical relaxation time in a metal can be so fast and why the speed of light is never exceeded.

D. Gauss's Law For a Metal Wire

When considering charge decay in a metal it is always helpful to have a picture in one's mind as to what the system might look like. Although the equation of charge decay (5) should be good for any shaped metal, calculations in this paper will use a copper metal wire as an example situation. The metal wire is a long cylinder of some specified radius and length. In order not to worry about charges moving in the z direction, let the wire be of such a significantly long length that the perturbing free-electron charges move essentially radially outward to the surface.

Consider, as shown in Fig. 1a, a Gaussian surface surrounding the center of a long conductor wire. If there is a uniform distribution of charge within the conductor, then the Gaussian surface will surround an amount of charge Δq_G as depicted in Fig. 1b. Gauss's law gives the amount of the charge enclosed within any incremental length dz as

$$\Delta q_G = \varepsilon_m E_r 2\pi r\, dz \tag{26}$$

where r is the radius of the Gaussian surface and E_r is the radial field at the Gaussian surface – Fig. 1a. This radial field E_r causes charges outside the Gaussian surface to move radially outward giving rise to a outward charge flow towards the surface of the wire.

So every unit area – each unit area surrounding a point on the Gaussian surface – will experience a current per unit area $\mathbf{J}_r$ flowing outward. This is depicted in Fig. 1b.

The volume charge density ρ_G within this Gaussian surface is by definition the amount

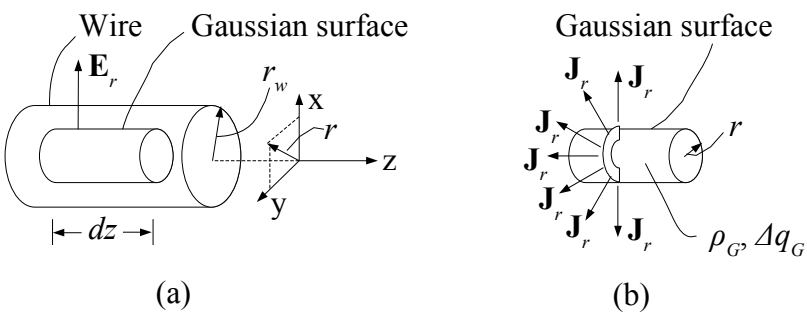

Fig. 1: Diagram of a wire with an incremental Gaussian surface drawn within the wire which encloses an amount of uniform charge Δq_G such that it has volume charge density ρ_G.

of incremental charge within the Gaussian volume divided by this incremental volume, or

$$\rho_G = \Delta q_G / (\pi r^2 dz) \tag{27}$$

so, equating the amount of Gaussian charge Δq_G within the surface based on (26) and (27) $\rho_G \pi r^2 dz = \varepsilon_m E_r 2\pi r dz$ gives the charge density within the Gaussian surface as

$$\rho_G = 2\varepsilon_m E_r / r \quad . \tag{28}$$

Hence, from (28) the electric field at Gaussian surface is

$$E_r = \rho_G r / (2\varepsilon_m) \tag{29}$$

and this radial E-field increases linearly with the radius of the Gaussian surface. This result will be used later to determine how much charge can be placed in a conductor before the E-field becomes so large that the free-electron mobility becomes dependent upon the magnitude of the E-field. But first, to understand how the E-field affects mobility requires an understanding of how the mobility and other important terms are obtained from collision averaging.

E. Averages

In order to proceed further in the understanding of when Ohm's law can and cannot be used, it is necessary to look more closely at why the electrical conductivity of a metal is a constant. The answer will be that the measurement of the electrical conductivity is done with an electric field that keeps the drift speed of the free-electrons quite low compared to the thermal speed of the free-electrons, and this in turn keeps the average speed of the free-electrons a constant. To understand this in detail, it is important to understand the calculation of averages.

Once it is realized that the average speed of the free-electrons is a constant – independent of the E-field when in the presence of a low electric field – the question can then be asked as to how large an electric field is too large. In studying charge decay, it is the perturbing charges placed inside the metal conductor that create the electric field which drives free-electron charges to the surface. The question that will be asked, and answered, is how much charge can be put into a metal conductor before the average speed of the free-electrons start to be dependent on the perturbing charges. Once this occurs the electrical conductivity will no longer be constant, and the exponential charge decay given by (5) or (21) will no longer be valid. So, conversely, there is a maximum charge density limit restriction that must be placed on the perturbing charge density ρ_0 in (5) – which is ρ_{p0} in (21) – above which Ohm's law is invalid and cannot be used in deriving (5) or (21).

1) General time average

In the appendix of [11] it was shown that, if t_0 is chosen as the zero of time ($t_0 = 0$) – that time being the end of the last collision and hence the beginning of time for a collision under study – the time average value of any species function Y over one collision by all the particles of that species is $\langle Y \rangle_i = \tau_{ci}^{-1} \int_0^\infty Y e^{-t/\tau_{ci}} dt$. However, $\int_0^\infty t^n e^{-t/\tau_{ci}} dt = n! \tau_{ci}^{n+1}$ so

$$\langle t^n \rangle = \tau_{ci}^{-1} \int_0^\infty t^n e^{-t/\tau_{ci}} dt = n! \tau_{ci}^n \quad . \tag{30}$$

With (30) it is relatively simple to obtain averages on a particular charged species during a contiguous collision averaging event.

2) Equations of motion for a free-electron

The equations of motion for a free-electron in the presence of an electric field **E** can be written down. If gravity can be ignored and if one direction of motion is taken to be in the direction of the electric field **E**, the equation of motion in the direction of **E** is

$$\mathbf{F}_E = m_e d\,\mathbf{v}_E / dt = s_e q_0 \mathbf{E} \quad . \tag{31}$$

Assume **E** may vary with position and time but is constant within the incremental region of this free-electron's travel during the short time $t \le t_{ce}$ with t_{ce} being the time at which the free-electron will experience a collision. For this constant electric field, integrating (31) gives the velocity in the E-field direction at any time t before a collision as

$$\mathbf{v}_E = \mathbf{v}_{E0} + (s_e q_0 / m_e) \mathbf{E} t \tag{32}$$

where $\mathbf{v}_{E0}$ is the velocity at the end of the previous collision (which is also the velocity of the start of the present collision under study). However, if – for example – the electric field is taken to be in the positive x-direction, this velocity can be written as

$$d\,\mathbf{x} / dt = \mathbf{v}_{E0} + (s_e q_0 / m_e) \mathbf{E} t \tag{33}$$

which after integration gives the position of the free-electron (relative to a fixed reference position $\mathbf{x}_0$ at the start of the upcoming collision) as

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{v}_{E0} t + [s_e q_0 / (2 m_e)] \mathbf{E} t^2 \tag{34}$$

3) Drift velocity of electrons

The average velocity of the electrons in the direction of the electric field can be determined by averaging the motion of all the free-electrons during a single collision. Time averaging (32) over all the free-electrons during a single collision gives

$$\langle \mathbf{v}_E \rangle = \langle \mathbf{v}_{E0} \rangle + (s_e q_0 / m_e) \mathbf{E} \langle t \rangle \tag{35}$$

But the averaging over $\mathbf{v}_{E0}$ will be zero since the initial velocity is as likely to be in the negative direction as it will be in the positive direction. So (30) into (35) gives

$$\mathbf{v}_d = \langle \mathbf{v}_E \rangle = (s_e q_0 / m_e) \mathbf{E} \tau_{ce} \quad . \tag{36}$$

as the drift velocity of the electrons due to the electric field. It can be seen that (36) is in agreement with the basic definition of the mobility for free-electrons as expressed by the combination of (12) and (13); namely,

$$\mathbf{v}_d = s_e b_e \mathbf{E} \tag{37}$$

which, since s_e = -1, shows the drift velocity of the free-electrons is opposite to the direction of – but proportional to – the electric field **E**, with the mobility b_e of the free-electrons being the constant of proportionality. Later it will be shown that above a certain electric field the collision time becomes dependent on the electric field so the mobility is no longer constant and becomes a function of the electric field.

4) Average speed of electrons

From (32) the square of the speed of a free-electron in the direction of the electric field is

$$v_E^2 = v_{0E}^2 + 2 v_{0E} (s_e q_0 / m_e) E t + [(s_e q_0 / m_e) E]^2 t^2 \tag{38}$$

and taking the average for all free-electrons over one collision in the field direction allows (38) to be written as the mean square speed

$$\langle v_E^2 \rangle = \langle v_{0E}^2 \rangle + 2 (s_e q_0 / m_e) E \langle v_{0E} t \rangle + [(s_e q_0 / m_e) E]^2 \langle t^2 \rangle \quad . \tag{39}$$

Here again the averaging over $v_{0E}\, t$ in (39) will be zero since the initial velocity will as likely be in the negative direction as it will be in the positive direction for any free-electron. Therefore, with (30) into (39) the mean square speed in the field direction becomes

$$\langle v_E^2 \rangle = \langle v_{0E}^2 \rangle + [(s_e q_0 / m_e) E]^2 2 \tau_{ce}^2 \quad . \tag{40}$$

By definition the mean square speed in three orthogonal coordinates is simply the sum of the squares in each of the coordinates. Without loss of generality the square of the speed of a free-electron in a rectangular coordinate system is given by $v^2 = v_x^2 + v_y^2 + v_z^2$; so, the average over all free-electrons during one collision will be given by the mean square speed $\langle v^2 \rangle = \langle v_x^2 \rangle + \langle v_y^2 \rangle + \langle v_z^2 \rangle$. Now, if the x-direction is taken to be that of the electric field, then using (40) as the *x*-direction the mean square of the speed is given by $\langle v^2 \rangle = \langle v_{0x}^2 \rangle + [(s_e q_0 / m_e) E]^2 2 \tau_{ce}^2 + \langle v_{0y}^2 \rangle + \langle v_{0z}^2 \rangle$. However, from the Maxwellian distribution of speeds in classical physics, when no *E* field is present, this mean square speed is [13] $\langle v^2 \rangle_{E=0} = \langle v_{0x}^2 \rangle + \langle v_{0y}^2 \rangle + \langle v_{0z}^2 \rangle = 3\, kT / m_e$ so the mean square speed with the field is $\langle v^2 \rangle = 3\, kT / (m_e) + [(s_e q_0 / m_e) E]^2 2 \tau_{ce}^2$. For a Maxwellian distribution of speeds, and as discussed in [13], the average speed $\langle v \rangle$ is related to $\langle v^2 \rangle$ and given by $\langle v \rangle = \sqrt{8/(3\pi)} \sqrt{\langle v^2 \rangle}$. Hence $\langle v \rangle = \sqrt{8\mathrm{kT}/(\pi m_e) + 16/(3\pi) [(s_e q_0 / m_e) E]^2 \tau_{ce}^2}$ is the average speed of the free-electrons. Thus, the average speed of the free-electrons can be written as

$$\langle v \rangle = \sqrt{v_{th}^2 + 16/(3\pi) v_d^2} \tag{41}$$

where

$$v_{th} = \sqrt{8\mathrm{kT}/(\pi m_e)} \tag{42}$$

is the thermal speed of the free-electrons and v_d is the drift speed of the free-electrons given by (36) or (37). Later it will be shown that to have a constant mobility and a constant electrical conductivity requires $\langle v \rangle$ to be constant, and hence (41) requires $v_d \ll v_{th}$. But from (36) or (37) this restricts the allowable maximum E-field.

5) Distance traveled per collision in direction of E-field

Averaging (34) gives the distance traveled in the field direction due to a collision as $\boldsymbol{\delta}_E = \langle \mathbf{x} - \mathbf{x}_0 \rangle = \langle \mathbf{v}_{E0} t \rangle + [s_e q_0/(2 m_e)] \mathbf{E} \langle t^2 \rangle$. Here again, since the initial velocity will as likely be in the negative direction as it will be in the positive direction at any time t, the averaging over v_{E0} t will be zero With the aid of (30) this averaging gives $\boldsymbol{\delta}_E = [s_e q_0/(2 m_e)] \mathbf{E} 2 \tau_{ce}^2 = [s_e q_0 \tau_{ce}/(m_e)] \mathbf{E} \tau_{ce}$ which can also be written as

$$\boldsymbol{\delta}_E = \tau_{ce} \mathbf{v}_d \tag{43}$$

Note: This is not the mean free path. This is just the average distance traveled in the field direction during the average collision time.

6) Mean free path

In the contiguous collision averaging method all the particles of a given species are watched during one collision and the average values of their motions are calculated. If the average speed $\langle v \rangle_i$ of the ith species is multiplied by the average collision time τ_{ci} of the species, it gives the average distance λ_i traveled in a collision. This average is called the mean free path λ_i of the species.

The mean free path should be only dependent on the structure of the material and the structure of the species. If r_a is the radius of an atom and r_i is the radius of the colliding ionic species under study, then for atoms of fixed centers, a particle of the ith species traveling at an average speed $\langle v \rangle_i$ will move a distance $d\lambda$ in time dt.

For hard sphere collisions, if an ith species particle approaches an atom along a line parallel to a center-line which goes through the center of the atom, the ith species particle will collide with the atom provided the ith species particle is located within a radius of $r_a + r_i$ from this center-line. The circular area over which a collision will occur is given by [13]

$$Q_i = \pi (r_a + r_i)^2 \tag{44}$$

and is referred to by various names the most common being the hard sphere collision-cross-section area, the cross-section for momentum transfer or the cross-section of interaction . The average particle of the ith species will travel a distance λ_i in a time τ_i and will sweep out a volume of $\lambda_i Q_i$ in time τ_{ci}. In other words, it will sweep out a volume on a per average collision time basis τ_{ci} of $\lambda_i Q_i / \tau_{ci} = \langle v \rangle_i Q_i$.

Let the atom that acts as the scattering site in a collision occupy the volume $\breve{V}$ during collision time τ_i. Then, for a general material containing N atoms in its volume V_N, this collision volume can be written as $\breve{V} = N \breve{V} / N = V_N / N = 1/(N/V_N) = 1/n_a$ where n_a is the number density of the atoms. Thus, the volume swept out per collision is $\breve{V}/\tau_i = 1/(n_a \tau_i)$. But this is the same as the volume swept out in the collision time τ_i and given above by $\lambda_i Q_i / \tau_{ci} = \langle v \rangle_i Q_i$. Equating these two volumes per collision time gives $\lambda_i Q_i / \tau_{ci} = \langle v \rangle_i Q_i = 1/(n_a \tau_{ci})$ and solving for the mean free path gives

$$\lambda_i = \langle v \rangle_i \tau_{ci} = 1/(Q_i n_a) \tag{45}$$

or conversely, solving for the collision time gives

$$\tau_{ci} = \lambda_i / \langle v \rangle_i = 1/(\langle v \rangle_i Q_i n_a) \tag{46}$$

If free-electrons are the colliding species, then $r_e << r_a$ and Q_i reduces to πr_a^2.

F. Calculations

Now that important averages have been done for the contiguous collision averaging method and, most importantly, now that the average speed (41) is known, it is possible to get a good understanding as to the physics involved in charge decay. Again, to make the problem real, a copper wire will be used as the example. The circuit form of Ohm's law has been used by others in the past to measure the electrical conductivity of copper. Knowing the conductivity and the charge density of the free-electrons in copper allows the electrical mobility of the free-electrons to be calculated. Then, knowing the mobility, the average collision time of the free-electrons can be calculated.

1) Mobility of free electrons in copper

From (19) and (25) the metal's conductivity is $\sigma_m = s_e^2 q_0 n_e b_e = q_0 (f_e n_a) b_e$ and is a measured quantity whose value for copper is given in Table 2. On the other hand, the mobility is a deduced quantity based on the measured conductivity and the assumption in (25) that for metals ($f_e = 1$). Assuming $f_e = 1$, and for the value of q_0 in Table 1 and the atom number density in Table 2, the mobility of the free-electrons in copper is

$$b_e = \sigma_m / (q_0 n_e) = 4.21 \, x \, 10^{-3} \, \mathrm{m^2/Vs} \quad .$$

2) Electron collision time in copper

With the the above calculated mobility the collision time for electrons in copper from (12) is $\tau_{ce} = b_e m_e / q_0 = 2.4 \, x \, 10^{-14} \mathrm{s}$. Since the electrical relaxation time τ_m of copper is 1.5 x 10^{-19} sec, and is orders of magnitude less that the collision time, it is clear that the controlling interest in electrical relaxation is not collisions but rather the electric field E supplied by the perturbing charges. This electric field is due entirely to coulomb repulsion. However, it should now be clear from (32) and the discussion of averages that the electric field supplied by the perturbing charges not only influences the perturbing charges, it also influence all the free-electrons normally found in the metal.

3) When mobility is and is not constant

The mobility of any charge species is a constant provided all values in (12) are constants. This means the mobility of the species will be a constant provided the collision time τ_{ci} of the species is a constant. But the collision time τ_{ci} is related to the mean free path λ_i through (45) and since λ_i is a constant, the mobility of any species can be written as

$$b_i = q_i / (m_i Q_i n_a \langle v \rangle_i) \tag{47}$$

or in terms of the thermal and drift speed as

$$b_i = q_i / \left(m_i Q_i n_a \sqrt{v_{th_i}^2 + 16/(3\pi) v_{d_i}^2} \right) \quad . \tag{48}$$

As long as the species' drift speed (see (37); replace subscript e by i in its development)

$$v_{d_i} = s_i b_i E \tag{49}$$

is very much less than the species' thermal speed

$$v_{th_i} = \sqrt{8\mathrm{kT}/(\pi m_i)} \quad , \tag{50}$$

then, with the restriction $v_{d_i} << v_{th_i}$, the mobility of the species reduces to

$$b_i = q_i/(m_i Q_i n_a v_{th_i}) \qquad b_i \text{ is a constant } (v_{di} << v_{th}) \tag{51}$$

On the other hand, if the electric field is such that the drift speed becomes a reasonable portion of the thermal speed both v_{th_i} and v_{d_i} must be retained and the mobility (48) written in terms of the E-field by using (49) becomes $b_i = q_i/\left(m_i Q_i n_a \sqrt{v_{th_i}^2 + 16/(3\pi) b_i^2 E^2}\right)$. Taking the square of both sides of this equation and then factoring out the mobility leaves a quadratic in b_i^2 with the solution

$$b_i = \sqrt{\frac{\left(\sqrt{1+[4^3/(3\pi)][q_i/(m_i Q_i n_a)]^2[E^2/v_{th_i}^4]} - 1\right)}{[(2)4^2 E^2/(3\pi)]} v_{th_i}^2} \quad . \tag{52}$$

For a small electric field the mobility (52) reduces to a constant as given by (51).

Thus, for a general material (solid, liquid or gas) to be ohmic, it must have a well defined and constant conductivity. Its conductivity is given by (17) where each species has a conductivity σ_i given by (9); and, as can be seen in (9), the species conductivity depends on its species mobility b_i. As a result, for a general material to have a constant electrical conductivity requires each species to have a constant mobility. Finally, a constant mobility is only achieved with the restriction $v_{d_i} << v_{th_i}$ for each species; but, if this restriction is met, the mobility of each species will be given by (51). For a metal conductor the free-electrons are the one and only mobile species; so, the restriction simply becomes a restriction on the free-electrons $v_{d_e} << v_{th_e}$.

4) Ohm's law drift speed restriction in metals

As discussed in the last section, the mobility of a species deviates from being constant when the average velocity of the species deviates from its thermal velocity. This deviation only occurs when an electric field is so strong as to cause the species drift velocity to become more than insignificant when compared to the species thermal velocity.

It is possible to set an arbitrary limit of say a 1% variation in mobility as the point where a further increase would mark the mobility as having changed too much to consider it a constant. From (47) it can be seen that a 1% deviation from a constant value would be $\langle v \rangle_i \leq 1.01 v_{th_i}$ or $\langle v \rangle_i^2 \leq 1.01^2 v_{th_i}^2$. With (41) this occurs when $v_{th_i}^2 + 16/(3\pi) v_{d_i}^2 = 1.01^2 v_{th_i}^2$ which can be solved for the maximum drift speed as $v_{d_i}^2 = (3\pi)(1.01^2 - 1)/16 v_{th_i}^2 = 1.18\, x\, 10^{-2} v_{th_i}^2$ or

$$v_{d_i} = 0.109\, v_{th_i} \qquad [\text{Max } v_{d_i} \text{ – ohmic restriction}]. \tag{53}$$

From (42) the thermal speed of electrons is $\mathbf{v}_{th_e}$ = 1.06 x 10^5 m/s at 293 K. So (53) gives a maximum electron drift velocity $\mathbf{v}_{d_e}$ of 1.15 x 10^4 m/s at 293 K. In a metal only

free-electrons contribute to conduction; so, for any metal, Ohm's law will no longer be valid above this drift velocity.

5) Ohm's law E-field restriction in metals

From (37) and (53) the magnitude of the electric field restriction on any species is

$$|E| \le |0.109\, v_{th_i} / (s_i b_i)| \qquad \text{[Max E-field – ohmic restriction].} \qquad (54)$$

In a metal only electrons are free to move, so only electrons need to be considered in the E-field restriction. Using (19) to calculate the mobility of copper from its measured conductivity, and (42) to calculate the thermal speed of electrons and substituting these values into (54) gives the E-field restriction for copper as 2.7 MV/m at 293 K.

If a metal other than copper is considered, then the mobility of the metal would be obtained from the metal's measured conductivity. Hence, the E-field restriction (54) is dependent on the metal.

From the above it can be concluded that:

►**iv)** The E-field in a metal conductor must be kept below a value of about 10% of the ratio of the thermal speed of the free-electrons divided by the mobility of the free-electrons, the mobility having been computed from the measured conductivity of the metal. This restriction is independent of the size or shape of the conductor. If E-field restriction (54) cannot be met, Ohm's law cannot be used.

It is interesting to note that this E-field restriction for copper is also close in magnitude to where air reaches its breakdown of 3 MV/m. Maybe there is a connection, but for now it is just a curious coincidence and nothing more. However, it does suggest that a further analysis might give some greater insight into what happens in solids, liquids and gases at high fields. What is clear here is that copper becomes non-ohmic for an E-field near and above about 2.7 MV/m. Hence, Ohm's law cannot be used to calculate charge decay of a copper metal if insertion of the perturbing charges results in an E-field above 2.7 MV/m anywhere in the metal.

6) Perturbing number density restriction in metals

The next question to be asked is how does this E-field restriction affect the amount of uniform charge that can be placed inside a metal conducting wire such as copper? The answer is clear. If the amount of uniform charge placed inside the wire initially gives the conductor a uniform charge density ρ_{p0}, then any Gaussian surface drawn inside the conductor will initially house a charge density $\rho_G = \rho_{p0}$. Thus, from (29) to keep the E-field in the wire below some defined field $E_{r_{max}}$ restricts the perturbing charge density to

$$\rho_{p0} < (2\, \varepsilon_m E_{r_{max}}) / r \qquad (55)$$

and restricts the perturbing number density to

$$n_{p0} < (2\, s_p \varepsilon_m E_{r_{max}} / q_0) / r \qquad (56)$$

where s_p = -1 for free-electrons. The minus sign in (56) presents no problem because $\mathbf{E}_r = E_r \hat{\mathbf{r}}$ and the radial field E_r will be negative since it points radially inward for a wire containing perturbing free-electrons; so $s_p E_{r_{max}}$ will always be a positive number.

As an example, it was shown in the previous section that the absolute value of the max-

imum E-field for copper is 2.7 MV/m. So, for a 2 mm diameter copper wire the radial E-field maximum is $E_{r_{max}}$ = -2.7 MV/m. Hence, (55) restricts the perturbing charge density ρ_{p0} to below -4.78 x 10^{-2} C/m^3, and (56) restricts the perturbing number density to below 3 x 10^{17} electrons/m^3. This may seem like a very high number density, but compared to the free-electron number density for copper of 8.46 x 10^{28} electrons/m^3 it is very small. As a result, if the perturbing charges are free-electrons, then the ratio $\sigma_{p0} / \sigma_m = n_{p0}/n_e$ and (56) shows for copper this ratio will be no greater than ≈ 3 x 10^{-12} if Ohm's law is allowed. Hence, the assumption $1 + \sigma_p 0 / \sigma_m = 1 + n_{p0}/n_e \approx 1$ is clearly justified. This assumption will be used in the next section during the development of the relaxation current equation.

7) Relaxation current

If a uniform charge density is placed inside a good conductor, coulomb charge repulsion will occur. As depicted in Fig. 1b, the radial current flowing outward due to the perturbing charges) from an incremental length *dz* of a Gaussian surface of radius *r* is

$$\begin{aligned} I_r &= J_r A \\ &= (\sigma E_r)(2\pi r\, dz) \\ &= [(\sigma_p + \sigma_m)\rho_G r/(2\varepsilon_m)](2\pi r\, dz) \\ &= [(\sigma_p + \sigma_m)/(\varepsilon_m)]\rho_0 \frac{e^{-t/\tau_m}}{1 + \frac{\sigma_{p0}}{\sigma_m}[1 - e^{(-t/\tau_m)}]}(\pi r^2 dz) \quad . \qquad (57) \\ &= [(\sigma_m)/(\varepsilon_m)]\rho_0 e^{-t/\tau_m}(\pi r^2 dz) \\ &= [\rho_0(\pi r^2 dz)]\tau_m^{-1} e^{-t/\tau_m} \end{aligned}$$

In (57) the first step is simply the writing down of the relationship between the current *I* and the current density *J* at the Gaussian surface. Then, to obtain the second step (18) was inserted for J_r, and in the third step (20) was inserted for σ and (29) was inserted for E_r. Next, in step four (21) (with $\rho_{p0} = \rho_0$) was inserted for the Gaussian charge density ρ_G . To be exact σ_p should also have contained (21), but it was intensionally not inserted here (but remember its maximum value is σ_{p0}); so, the next step can be seen clearly. Namely, for a metal σ_{p0} is several orders of magnitude smaller than σ_m. So, with the assumption $1 + \sigma_{p0} / \sigma_m \approx 1$, step four becomes step five. The assumption $1 + \sigma_{p0} / \sigma_m \approx 1$ is also tantamount to recognizing that for a metal (21) reduces to (5). Finally, in the last step, step 6, the electrical relaxation time τ_m was inserted based on the definition as given by (6). The end result is (57) and gives the time dependence of the radial outward charge movement due to the presence of a perturbation charge density ρ_0 (or ρ_{p0}) at time $t = 0$ in the wire.

It is very important to the discussion of the Ohm's law paradox to understand what happened between step four and step five (between the fourth and fifth equal sign). In step four, if everything were multiplied out about the term $(\sigma_p + \sigma_m)$, there would be two currents I_p and I_m, with I_p associated with σ_p and I_m associated with σ_m. So, dropping σ_p because $\sigma_{p0} << \sigma_m$ means $I_p << I_m$; and, as a result, the current in (57) becomes just I_m. Hence, almost all the free-electron motion that makes up this current is due to the free-electrons in the ohmic conductor and an insignificant amount is due the free-electrons in-

serted as a perturbation. On the other hand, all of the driving force (the E-field) that causes the free-electron motion is due to the insertion of the perturbation charges.

The radial outward current in (57) is just a current which by definition is $I = dq/dt$. Thus, substituting this definition of current into (57) and setting up the integration gives

$$\int_{q_0=0}^{q_{rem}} dq = \rho_0(\pi r^2 dz)\int_0^t \tau_m^{-1} e^{-t/\tau_m} dt = -\rho_0(\pi r^2 dz)\int_0^t d\left(e^{-t/\tau_m}\right) \quad (58)$$

Performing the integration in (58) gives, as a function of time, the amount of charge removed from within the Gaussian surface as

$$q_{rem} = \rho_0(\pi r^2 dz)(1 - e^{-t/\tau_m}) \quad . \quad (59)$$

The charge density removed will be this charge divided by the incremental volume $\pi r^2 dz$ from which this charge has exited and is

$$\rho_{rem} = q_{rem}/(\pi r^2 dz) = \rho_0(1 - e^{-t/\tau_m}) \quad (60)$$

This clearly shows that as $t \to \infty$ the total charge ρ_0 is removed with most removed in $5\tau_m$.

8) Radial drift velocity of the free-electrons

The radial drift speed for free-electrons is given by (37) $v_{dr} = b_e E_r$ and following the same procedure as was used in (57) to replace the E-field with the time dependent charge decay (37) becomes

$$v_{dr} = \rho_0 b_e e^{-t/\tau_m} r/(2\varepsilon_m) \quad (61)$$

At first glance it appears (61) implies at $t = 0^-$ the drift velocity is zero because $\rho_0 = 0$, and at $t = 0^+$ the drift velocity jumps to its maximum drift and decays back to near zero in about $5\tau_m$. So, at first glance it would seem like (61) implies an almost infinite acceleration at $t = 0$. However, that is not the full story. Going back to the mind's-eye laboratory where any experimental procedure is possible, some elaborate injection device had to be conceived and used to add the perturbing free-electrons into the metal wire in the first place. In the mind's-eye bring $\rho_0 + \rho_{su}$ into the wire and hold the charges in place with the elaborate injection device. By holding both ρ_0 and the set-up charge density ρ_{su} in place, the drift velocity of all the charges is zero. Now release all the charges; they start moving and pick up velocity from their coulomb repulsion field with time as described by (32). After the charge density decays from $\rho_0 + \rho_{su}$ to just ρ_0 remove the charges that have already arrived at the surface of the wire and start the clock at $t = 0$. Now the set-up charge density is gone, and the initial drift velocity is that given by (61) at $t = 0$. In other words, it is appropriate to have an initial velocity at $t = 0$, because $t = 0$ was set as an arbitrary time in the decay process and not the specific time when there is zero initial velocity.

9) Radial movement of the free-electrons

By definition, the incremental radial movement is given by $dr = v_{dr}\, dt$; so, inserting (62) into this definition gives

$$dr = \rho_0 b_e e^{-t/\tau_m} r/(2\varepsilon_m) dt \quad (62)$$

and setting up the integration gives

$$\int_{r}^{r+\delta_r} dr/r=(\rho_0 b_e)/(2\varepsilon_m)\int_0^t e^{-t/\tau_m}dt=(1)/(2\tau_p)\int_0^t e^{-t/\tau_m}dt \qquad (63)$$

where at $t = 0$ charges on the Gaussian surface are at r and at $t > 0$ they have moved to a position at $r + \delta_r(t)$. The solution to (63) in terms of $\delta_r(t)$ is $\delta_r(t)=r\left[e^{\tau_m/(2\tau_p)(1-e^{-t}/\tau_m)}-1\right]$; but, if the perturbing charges are electrons, then $\tau_m/\tau_p=n_{p0}/n_e \ll 1$ and this solution reduces further to

$$\delta_r(t)\approx r\left[\tau_m/(2\tau_p)(1-e^{-t/\tau_m})\right] \qquad (64)$$

and the maximum distance the free-electrons travel ($t \to \infty$) will be

$$\delta_r=r\tau_m/(2\tau_p)=(r/2)(n_{p0}/n_e) \qquad (65)$$

As an example, look at a copper wire of 2 mm diameter with a maximum perturbation charge density based on the maximum E-field of 2.7 MV/m. For this situation it was previously shown that for copper the ratio n_{p0}/n_e will be ≈ 3 x 10^{-12}. At $r = r_w$ the Gaussian surface is set at the surface of the wire, and this is the surface where an amount of free-electrons – equal to the original amount of the perturbing free-electrons – ends up. For this situation, using (65) the distance the charges closest to the wire surface traveled is δ_r = 1.78 x 10^{-15} m.

Since the relaxation time is very much less than the collision time, it is the electric field of the perturbing free-electrons that drives the drift of all the free-electrons in the metal. Hence, a very rough estimate of the distance δ_r can also be made by taking the average drift velocity and multiplying it by the average relaxation time (rather than the average collision time) $\delta_r \approx v_{de}\tau_m$ and this calculation gives δ_r = 1.72 x 10^{-15} m. Although for the present calculation this distance is very close to that calculated using (65) it should be observed that (65) shows the distance traveled is a function of r; i.e., a function of the position of the charges. In other words, charges near the center of the wire drift a much shorter distance than charges near the surface of the wire, and that is why $\delta_r \approx v_{de}\tau_m$ gives only a very rough estimate.

It is also interesting to compare the distance δ_r to the classical radius of an electron (r_e = 2.82 x 10^{-15} m) which suggests free-electrons move less than a diameter's distance from their normal positions in order to accomplish charge decay. The finite size of the electron also suggests that when the free-electrons reach the surface they are situated in a surface having a thickness δ_r as opposed to the usual assumption of a zero thickness interface.

10) Surface layer thickness

To pursue the idea of a finite thickness interface further, assume the surface has a volume that stores these charges that arrive at the surface. If so, then the incremental volume at the surface of the wire where these charges end up would be $\acute{V}_s=\pi(r_w+\delta_r)^2 dz-\pi r_w^2 dz \approx 2\pi\delta_r r_w dz$. This volume must hold all the free-electron charges that arrive. The amount of charge that arrives at the wire surface is given by (59) with $r = r_w$; and, when all the charges have arrived, there will be a total amount of charge

given by $q_{rem}=\rho_0(\pi r_w^2 dz)$. These charges will be in the volume $\acute{V}_s$ calculated above so the volume charge density of the free-electrons on the surface can be defined as

$$\rho'_{sv}=q_{rem}/\acute{V}_s=\rho_0(\pi r_w^2 dz)/(2\pi\delta_r r_w dz)=\rho_0 r_w/(2\delta_r) \tag{66}$$

and is the charge per unit volume within the thickness layer δ_r. Note in Fig. 1 that if a Gaussian surface is placed at the surface of the wire the charge per incremental volume is $\rho_0=q_{dz}/(\pi r_w^2 dz)$, the charge per incremental surface is $\rho_s=q_{dz}/(2\pi r_w dz)$, and the ratio is $\rho_s/\rho_0=r_w/2$; so,

$$\rho_s=\rho_0 r_w/2 \tag{67}$$

is the surface charge density at the wire after the charges have decayed from within the body of the wire. If this surface charge were actually in a thickness layer δ_r at the surface, then the volume charge density (of this surface charge density) would be

$$\rho'_{sv}=\rho_s/\delta_r=\rho_0 r_w/(2\delta_r) \tag{68}$$

in agreement with the definition (66) above. According to (65) at the wire $\delta_r=(r_w/2)(n_{p0}/n_e)=(r_w/2)(\rho_0/\rho_e)$ and from (68) this gives

$$\rho'_{sv}=\rho_e \tag{69}$$

as the effective volume charge density of the charges at the surface.

Equation (69) gives a most interesting result. It says the charges near the wire surface move out into a volume at the surface such that when the decay is finished, these charges will occupy a volume equal to the volume charge density of the free-electrons in the metal, the thickness of this volume being adjusted such that it will accommodate all the perturbation charges, no matter the initial magnitude of the perturbation charge density ρ_0.

With the ion cores of the atoms having a volume charge density of ρ_{a+} the equilibrium volume charge density in the bulk is $\rho = \rho_{a+} + \rho_e = 0$ and at the surface is $\rho'_{sv}=\rho_e=-\rho_{a+}$. This exact match of the volume charge density at the surface to the volume charge density of the free-electrons (and ion core atoms) in the bulk suggests some sort of coulomb alignment of the charges at the surface occurs to accommodate the positions of the free-electrons and ion-cores of the atoms in the bulk conductor. The result makes one wonder if there should not be more research in interface thickness!

III. Conclusions

It was Maxwell [1] who pointed out Ohm's law (1) is only valid for a homogeneous conductor and only valid when the conductor's electrical conductivity σ_m is a constant and not a function of the potential applied across the conductor or a function of the current that passes through it. In this paper a detailed look at the conductivity revealed that when perturbing charges are placed uniformly inside a good conductor that these charges create an electric field, and the larger the size of the conductor the larger will be the field as one moves radially outward. It was also shown that if the field is too large – in excess of a value E_{max} – it will affect the electrical conductivity σ_m and negate the use of Ohm's law.

It was further shown that E_{max} can be calculated using (54) for any metal and is dependent on the measured conductivity (or inferred mobility of the free-electrons) of the metal. A discussion on the make-up of conductivity showed only free-electrons can be inserted into an isolated metal conductor as any other charging changes its thermodynamic equilibrium conductivity. It was further noted that for perturbing charges the equation for charge decay is given by (21), but for metals, where the perturbing charges are restricted to free-electrons, it reduces to (23) and then – due to the E_{max} restriction – reduces further to the well known decay (5) found in textbooks. Finally, charge decay of free-electrons based on (5) was examined in a copper wire and the result showed the radial charge movement to the wire surface is due to movement of all the free-electrons and not just the perturbing free-electrons. The examination further showed within the short time period of an electrical relaxation time (6) the free-electrons only drift a fraction of a free-electron diameter. It is found that the free-electrons next to the surface of a wire contribute to the current that builds up the free-electrons at the wire surface as the internal perturbation charge density decays into the metal's free-electron charge density. The end result is a perturbing free-electron near the center of a conductor does not have to travel to the surface of the conductor, so no free-electron ever needs to travel faster than the speed of light. Essentially, the perturbing free-electrons create the electric field as the driving force, but it is the free-electrons normally found in the metal conductor that are the main players in controlling the decay. The total decay process can be understood based on Ohm's law (1) with the Ohm's law solution for charge decay given by (5) and with the time constant given by (6). Consequently, the Ohm's law paradox does not exist, and the results presented here should end any further discussion of the paradox.

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