# Electrostatic force between two conducting equal-sized charged spheres

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*Abstract*—**We analyze the electrostatic force between two equal-sized charged conducting spheres. We provide exact closed-form expressions for the force in terms of the special q-digamma function. Additionally, we provide simpler-to-use approximate expressions for the force and compare the approximations to the exact results.**

## I. Introduction

The setup of the problem we consider is shown in Fig. 1. Two conducting spheres $A$ and $B$ of radii $R$ with a center-to-center distance $S$ between them, are being held at voltages $V_a$ and $V_b$ respectively. The permittivity of the surrounding medium is $\varepsilon$.

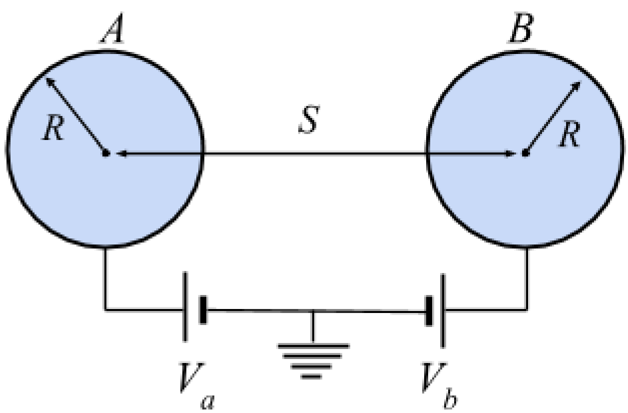


Fig.1. Basic geometry of our setup. The charge on each sphere depends upon the applied voltages $V_a$ and $V_b$, and the distance $S$ between them.

One way to calculate the capacitance coefficients, $C_{aa}$ and $C_{ab}$, is by using the method of images, which yields the well known result [1, 2]

$$c_{aa} \equiv \frac{C_{aa}}{4\pi\varepsilon R} = \sinh\beta \sum_{n=1}^{\infty} \operatorname{cosech}[(2n-1)\beta] \,,$$
$$c_{ab} \equiv \frac{C_{ab}}{4\pi\varepsilon R} = -\sinh\beta \sum_{n=1}^{\infty} \operatorname{cosech}[2n\beta] \,, \qquad (1)$$

where $\beta = \cosh^{-1}[S/2R]$ and $c_{aa}$ and $c_{ab}$, are the dimensionless versions of $C_{aa}$ and $C_{ab}$.

The capacitance solutions in Equation 1 were numerically calculated by Pisler and Adhikari accurately to one part in a million [3]. As the separation between the spheres decreases higher and higher number of terms are needed to achieve any fixed desired accuracy.

## II. Exact capacitance coefficients in closed form

In a recent paper [4] we presented exact closed-form expressions for the capacitance coefficients by summing the infinite series in Equation 1. The sum were calculated by using the substitution $x = s - \sqrt{s^2 - 1}$ where $s \equiv S/2R$ is the dimensionless center-to-center distance between the two spheres. Here we present the capacitances calculated in [4] in a more compact form:

$$c_{aa} = \left(\frac{1-x^2}{x}\right) \frac{2\psi_{x^2}(1/2) - \psi_{x^4}(1/2) + \log\left(\frac{1-x^2}{1+x^2}\right)}{4\log x} \,,$$
$$c_{ab} = -\left(\frac{1-x^2}{x}\right) \frac{\psi_{x^4}(1/2) + \log(1-x^4)}{4\log x} \,, \qquad (2)$$

where $\psi_q(z)$ is the q-analog of the digamma function of z. In the capacitance expressions above $q = x$ and $z = 1/2$.

The capacitance expressions in Equation 2 are closely related to the exact result for the capacitance between a sphere and infinitely grounded plane [5]. Here we reexpress the sphere-plane capacitance in a more compact form:

$$c = \left(\frac{1-x^2}{x}\right) \frac{\psi_{x^2}(1/2) + \log(1-x^2)}{2\log x} \,. \qquad (3)$$

Using Equations 2 and 3 it can be show that the capacitor of the the two equal-sized sphere system can be written in terms of the sphere-plane capacitance in the following manner

$$c_{aa} = c - \frac{x}{1+x^2} c(x^2) \,,$$
$$c_{ab} = -\frac{x}{1+x^2} c(x^2) \,, \qquad (4)$$

where $c(x^2)$ is the expression for $c$ in Equation 3 but with $x$ replaced by $x^2$.

## III. Approximate capacitance coefficients

The capacitance results in Equation 2 involve the q-digamma function. This special function is not available in most mathematical software and spreadsheets. Hence, there is a need for accurate approximations that use simpler functions.

Since both capacitance coefficients of the two-sphere system can be expressed in terms of the sphere-plane capacitance through Equation 4, we first approximate the sphere-plane capacitance $c$ with

$$\tilde{c}_0 = \frac{1+x+3x^2+x^3+3x^4+x^5+x^6}{1+2x^2+2x^4+x^6} - \frac{(1-x^2)\tanh^{-1}x^{5/2}}{x\,\log x} \;. \tag{5}$$

The above expression approximates $c$ to within 0.3% at all distances. We improve this approximation by calculating the difference in capacitance between $c$ and $\tilde{c}_0$ to order $(x-1)^2$ to give the second order correction

$$\Delta\tilde{c}_2 = \left[\gamma + \log\tfrac{7}{2} - \tfrac{11}{6} + \left(\tfrac{11}{3}\gamma + \tfrac{11}{3}\log\tfrac{7}{2} - \tfrac{323}{48}\right)(x-1)^2\right] e^{-\frac{7}{2}\tanh^{-1}(1-x^2)}, \tag{6}$$

where $\gamma = 0.5772...$ is the Euler Gamma constant. The approximate capacitance for the sphere-plane system can now be written as:

$$\tilde{c}_2 = \tilde{c}_0 + \Delta\tilde{c}_2 \;. \tag{7}$$

The expression for $\tilde{c}_2$ approximates the exact capacitance in Equation 3 to within 0.03% at all distances.

At the cost of increasing its mathematical complexity, the capacitance approximation can be further improved by adding additional correction terms in higher powers of $(x-1)$. For example, adding the additional $(x-1)^3$ term to $\tilde{c}_2$ gives

$$\tilde{c}_3 = \tilde{c}_2 + \left(-\gamma + \tfrac{1}{6}\log\tfrac{7}{2} + \tfrac{5}{16}\right)(x-1)^3\, e^{-\frac{7}{2}\tanh^{-1}(1-x^2)}, \tag{8}$$

accurate to within 0.025% at all distances.

The approximate capacitances for the two sphere system $\tilde{c}_{aa}$ and $\tilde{c}_{ab}$ can be written in terms of the approximate capacitances $\tilde{c}_2$ or $\tilde{c}_3$ in the same fashion as in Equation 4. In Fig. 2 below, the exact and approximate capacitance coefficients for the two sphere system are plotted together for comparison. The approximate capacitances plotted with dashed lines are virtually indistinguishable from the exact capacitances plotted with solid lines.

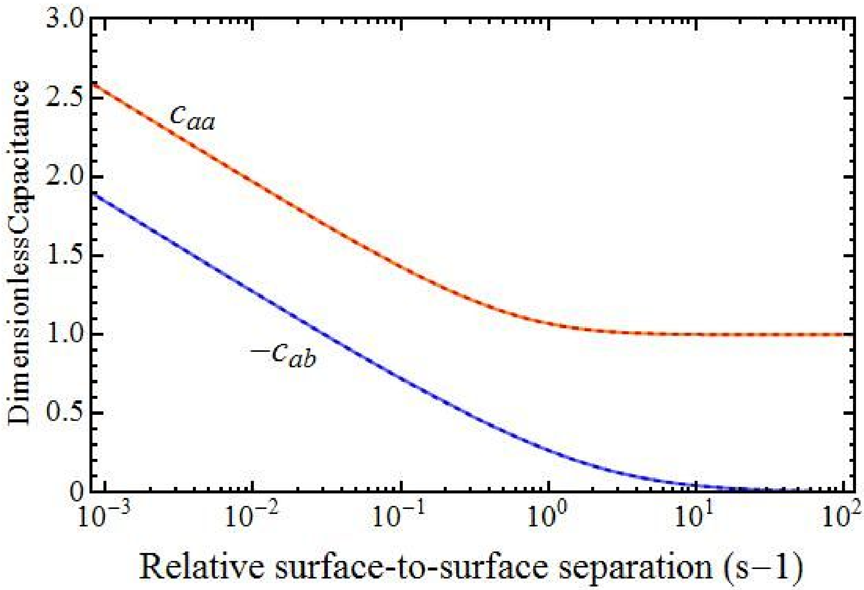


Fig. 2. The dimensionless capacitance coefficients for two equal-sized spheres are plotted versus their relative surface-to-surface separation $s-1$. The solid lines are the exact results in Equation 2 and the dashed lines are the approximations obtained by substituting Equation 7 into Equation 4. The relative error in the approximations is only 0.05% and the two approximation curves lie on top of their corresponding exact results.

## IV. Force between two spheres

The electrostatic energy of the two sphere system is given by [1, 2]

$$W_V = \tfrac{1}{2}C_{aa}\left(V_a^{\,2} + V_b^{\,2}\right) + C_{ab}V_aV_b. \qquad (9)$$

The electrostatic force between the two spheres is derivative of the electrostatic energy with respect to the sphere separation,

$$F_V = -\frac{dW_V}{dS} = -\frac{1}{2}\left(V_a^{\,2} + V_b^{\,2}\right)\frac{dC_{aa}}{dS} - V_aV_b\frac{dC_{ab}}{dS}. \qquad (10)$$

This force can be written in dimensionless form as

$$f_V \equiv \frac{F_V}{\pi\varepsilon V_a^{\,2}} = -(1+v_b^2)\frac{dc_{aa}}{ds} - 2v_b\frac{dc_{ab}}{ds}. \qquad (11)$$

The dimensionless force can be more easily calculated by expressing it in terms of $x$.

$$f_V = -\left(\frac{4x^2}{1-x^2}\right)\left[(1+v_b^2)\frac{dc_{aa}}{dx} + 2v_b\frac{dc_{ab}}{dx}\right]. \qquad (12)$$

In Fig. 3 below we plot the force between the two spheres versus their relative surface-to-surface separation for several voltage ratios $v_b$. Also plotted are the approximate forces obtained by using the two approximations in Equations 7 and 8. Our results agree with Lekner's result [6] that the spheres always attract each other provided they are sufficiently close, unless they have the exact same voltage.

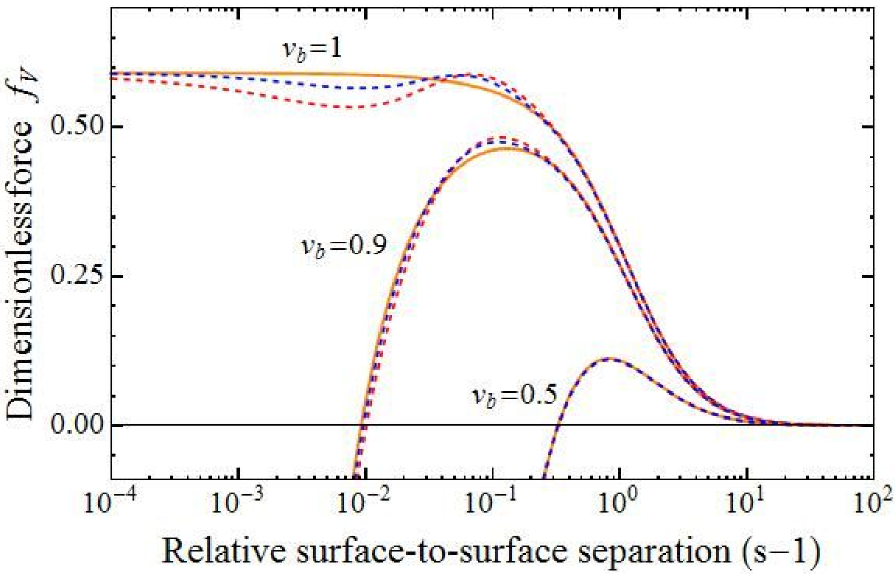


Fig. 3. The dimensionless force between two equal-sized spheres is plotted versus their relative surface-to-surface separation $s-1$ for several different voltage ratios. The solid lines are the exact results and the red and blue dashed lines are the approximations obtained by substituting Equations 7 and 8 respectively into Equation 4.

## V. Acknowledgements


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