TOWARDS THE KELVIN FORMULA OF FORCES ACTING ON POLARIZED BODIES

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***Abstract*—Electromagnetic forces provide us with various phenomena and mechanisms of key importance for multiple applications. Because of their importance, the key notions of electromagnetism should be elucidated with a maximum clarity. Among those notions of key importance are the forces, energy, and entropy of metallic and polarizable substances. Yet, these key notions remain unclear and, as such, repeatedly trigger multiple discrepancies and hot debates (see, for instance, [1--4])**

The simplest instrument of analysis of the resultant electrostatic forces acting on polarizable substances can be achieved with the help of the Kelvin's formula:

$$\boldsymbol{F}_{res} = \int_{\Omega} d\Omega P^i \nabla_i \boldsymbol{E}, \qquad \textbf{(*)}$$

where φ is the electrostatic potential, $\boldsymbol{P}$ is the polarization density, and the integration in (*) is taken over the whole polarized body. Similar relation is often use in magnetostatics also.

The elegant formula (*) is intuitively transparent and simple. Not surprisingly, it has become the "working horse" of multiple engineering applications and academic studies. It faces, however, some obstacles (see, the discussion in [4]). It was demonstrated more recently that it leads to some paradoxes of non-vanishing self-action [5,6]. We will discuss those paradoxes and show how they can be avoided based on the minimum energy approach.

I. SELF-ACTION OF DISTRIBUTED CHARGES

Consider a rigid body with distributed charges $q(z)$ within a finite domain ω surrounded by electrically neutral space. Then, the electrostatic potential φ is given by the bulk system:

$$\Delta\varphi = \begin{cases} -4\pi q(z), \text{within}\,\omega \\ 0, \text{otherwise} \end{cases} \tag{1}$$

amended with the boundary conditions

$$[\varphi] = \left[\nabla^i \varphi\right] N_i = 0 \tag{2}$$

and decay condition at infinity

$$\lim_{|z|\to\infty} \varphi(z) = 0 \tag{3}$$

where $[a]$ denotes the jump in the enclosed quantity across the boundary of ω. The ponderomotive force per unit volume is given by $\boldsymbol{f} = -q\nabla\varphi$ and the resulting net self-force $\boldsymbol{F}$ upon the domain ω is given by the integral $\boldsymbol{F} = \int_\omega d\omega \boldsymbol{f}$. It is easy to formally show that this force vanishes, consistent with a wide range of commonly observed phenomena. Indeed, the resultant force $\boldsymbol{F}$ is also be presented by the repeated integral

$$\boldsymbol{F} = \int_\omega d\omega \int_\omega d\omega * \frac{q(z*)q(z)}{|\boldsymbol{z}-\boldsymbol{z}^*|^3}(\boldsymbol{z}-\boldsymbol{z}^*) \tag{4}$$

We can demonstrate with the help of (4) that the resultant force vanishes. Indeed, after a double change $z^i \leftrightarrow z^{i*}$ in (4) we arrive at the relation $\boldsymbol{F} = -\boldsymbol{F}$ which is equivalent to vanishing of the resultant force

$$\boldsymbol{F} = 0 \tag{5}$$

One might say that validity of the relationship (5) is obvious straightforward based on the third Newton's law, ingrained into the Coulon law of electrostatics. Alternatively, one can say that it is obvious from our everyday experience. If the relationship (5) were violated the rigid body with ingrained electric charges in it would be self-accelerating without any external forces whatsoever. These two arguments are very strong but insufficient. Indeed, the system of electrostatics (1) - (4), though motivated by the Coulon's law includes also the procedure of homogenization. This procedure is no way obvious or straightforward. For instance, we are well aware that the electrostatic energy of the system of the system discrete electric points charges can be positive or negative; for the same token, the electrostatic energy of a an isolated point charge is infinite. However, after the traditional homogenization of the Coulon law the electrostatic energy of any spatially limited distribution of charges with finite density is always positive. In other words, the electrostatics of discrete electric charges, from one hand, and the electrostatics of distributed charge have not only multiple common features but also some qualitative distinctions. Thus, the above proof of vanishing of the resultant force of the distributed charges is in no way redundant.

II. SELF-ACTION OF DISTRIBUTED DIPOLES AND THE KELVIN'S FORMULA

Let us now consider a related problem in which the electric charge distribution $q(z)$ is replaced with the electric dipole distribution $\boldsymbol{P}(z)$. The induced electrostatic potential φ is governed by the system:

$$\Delta\varphi = \begin{cases} 4\pi\nabla_i P^i(z), \text{within } \omega \\ 0, \text{otherwise} \end{cases} \tag{6}$$

amended with the boundary conditions

$$[\varphi] = \left[\nabla^i\varphi - 4\pi P^i\right] N_i = 0 \tag{7}$$

and decay condition at infinity

$$\lim_{|z|\to\infty} \varphi(z) = 0 \tag{8}$$

The proper modeling of the resulting ponderomotive forces constitutes an important, challenging, and still-controversial open problem. A commonly accepted expression for the ponderomotive force per unit area is given by the Kelvin formula [7, p. 360], [8, p. 388], [9, p. 99]:

$$\boldsymbol{f} = P^i \nabla_i \boldsymbol{E} \tag{9}$$

and the resulting net force upon the domain ω is given by

$$\boldsymbol{F} = \int_\omega d\omega \boldsymbol{f} = \int_\omega d\omega P^i \nabla_i \boldsymbol{E} \tag{10}$$

We report that, unlike the force associated with a distribution of electrical charges and the vast experimental evidence, the force $\boldsymbol{F}$ given in equation (10) does not vanish, except for a limited number of special cases, such as a spherically symmetric distribution or a constant distribution in elliptical domains. The striking disagreement with the experimental evidence is the essence of the paradox reported in this note.

The non-vanishing self-force can be demonstrated in a number of ways. A detailed analytical derivation will be given in a forthcoming expanded paper. One of the simplest ways for computationally minded reader to see a convincing example is to compute a finite element solution for a spherical domain ω with non-constant .field $\boldsymbol{p}$, where the infinite surrounding space is approximated by a sufficiently large domain.

III. Thermodynamically consistent approach for deformable dielectrics

Determination of the resultant force is sufficient when dealing with rigid bodies. When dealing with deformable bodies we ought to amend the equations of electrostatics with analogy of equations of elasticity in which ponderomotive forces are taken into account. Consider a deformable polarizable substance in a thermostat maintained at fixed temperature T°. In what follows the parameter T° will be omitted from all the relationships. The free energy ψ per unit mass is given by the following formula

$$\psi = \psi(\nabla_i U_j, P^k)\,, \tag{11}$$

where P^k—the polarization vector, U_i—the displacement vector.

This substance reacts on the external load by generating elastic strain $\nabla_i U_j$ and electric polarization P^k.

At the internal and external boundaries these vectors should satisfy the following boundary conditions:

$$[\varphi]_-^+ = 0 \tag{12}$$

and

$$\left[D^i\right]_-^+ N_i = 0\,, \tag{13}$$

where the displacement D^i is as always

$$D^i = E^i + 4\pi P^i\,. \tag{14}$$

The bulk equations of the electrostatic and mechanical equilibrium read

$$\rho \frac{\partial \psi}{\partial P^i} = E_i \tag{15}$$

and

$$\nabla_m \aleph^{mk} = f^k\,, \tag{16}$$

$$\left[\aleph^{mk}\right]_-^+ N_m = 0\,, \tag{17}$$

where f^k is the distributed force density, and the Aleph tensor $\aleph^{mk}$ is defined as follows:

$$\aleph^{mk} \equiv \rho \frac{\partial \psi}{\partial \nabla_m U_j}\left(\delta_j^k - \nabla_j U^k\right) - z_{\cdot\cdot}^{mk}\left(\frac{1}{4\pi} E_l D^l - \frac{1}{8\pi} E_l E^l\right) + \frac{1}{4\pi} D^m E^k \tag{18}$$

For the elastic dielectrics the Aleph tensor $\aleph^{mk}$ plays the role of stress tensor. With this understanding, using the relationships (16), (17), one can prove that the resultant self-action force vanishes.

A detailed discussion of the Aleph tensor $\aleph^{mk}$ can be found in [6].

IV. CONCLUSION

In conclusion, we presented a paradox of non-vanishing self-force associated with the classical Kelvin formula (9) for ponderomotive forces in polarized substances. This shows that utmost care and advanced physical intuition should be used when utilizing (9), and especially (10). Furthermore, it is desirable to come up with a ponderomotive force theory that is free of the described contradiction. One possible approach, rooted in the thermodynamic framework [2], [4], [6], will be proposed in the expanded paper.

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