

A New Consideration In Elementary Electrostatics

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Abstract— The problem of evaluation of the field of a surface charge $\sigma = \sigma_0 \cos \theta$ over a spherical surface centered at the origin, with θ being the polar angle, is quite well known in electrostatics. In addition, the naïve method of direct integration is far more difficult than those involving methods of vector calculus, and the solution is usually evaluated by first evaluating the potential by the solution of Laplace's equation. In 1992, however, David J. Griffiths introduced a new short method of solution in a short note entitled "The field of a uniformly polarized object". Here we express his work as a theorem which essentially talks of surface charges in place of polarized objects. We express the theorem in two forms and then use it to establish an illuminating theorem concerning the common enclosure of two arbitrary closed surfaces in space. Nevertheless, the major job of this paper is to explain a wonderful agreement between two ways of solution of the field of a uniformly polarized sphere.

I. INTRODUCTION

Perhaps one of the most common questions in electrostatics is this: what is the field of a uniformly polarized sphere of polarization \mathbf{P} ? There are two common ways to tackle the situation. One is to notice that the bound charges reside only on the surface. The surface charge density varies as $\mathbf{P} \cos \theta$, where θ is the angle between the direction of polarization and the radius vector relative to the centre of the sphere. The other way is to picture the process of polarization as a small stretching of the sphere. We say, there are two spheres of uniform charge densities $+\rho$ and $-\rho$, which coincide when there is no polarization; and when polarization occurs, the centre of positive charge moves by a distance d (in the direction of polarization) relative to the centre of negative charge. The field is evaluated as the net field of two uniformly charged spheres in the limit $d \rightarrow 0$, $\rho \rightarrow \infty$ while the product ρd held fixed at the value \mathbf{P} . The answers from these two ways of solutions match¹. In a state of mind placed in the framework of the process of polarization, no one asks why? But I believe that the question can be asked and even answered. This is the central theme of this paper.

II. A SWEET THEOREM IN ELECTROSTATICS

In a short note² published in 1992, David J. Griffiths introduced the concept that there exists a curious short method of solution of the field of a uniformly polarized object whenever it is easily possible to evaluate the field of a uniformly charged object of the same shape and size. We restate his work as a theorem here:

Theorem: Suppose that the surface charge density over a closed surface S varies as $\sigma = \sigma_0 \cos \theta$, where σ_0 is a positive constant and θ is the angle that the outward normal forms with the positive direction of the axis of z . Then the field \mathbf{E} of the surface charge equals $-\sigma_0 \partial \mathbf{E}_1 / \partial z$, where \mathbf{E}_1 is the field per unit charge density due to a uniform charge distribution in the region R enclosed by S .

We refer to this type of a charge distribution as a normal charge distribution. Our theorem switches the problem of evaluation of the field of a certain species of variable charge distribution over a closed surface to the evaluation of the field of a uniform charge distribution over a connected region. Naturally then, it can be best proved by using the divergence theorem.

The potential of the surface charge is

$$\begin{aligned}
 V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \iint \frac{\sigma_0 \cos \theta}{|\mathbf{r} - \mathbf{r}'|} da' \\
 &= \frac{1}{4\pi\epsilon_0} \iint \frac{\sigma_0 \hat{\mathbf{z}} \cdot d\mathbf{a}'}{|\mathbf{r} - \mathbf{r}'|} \\
 &= \frac{1}{4\pi\epsilon_0} \iiint \bar{\nabla}' \cdot \left(\frac{\sigma_0 \hat{\mathbf{z}}}{|\mathbf{r} - \mathbf{r}'|} \right) dv' \\
 &= \frac{1}{4\pi\epsilon_0} \iiint (\bar{\nabla}' \left(\frac{\sigma_0}{|\mathbf{r} - \mathbf{r}'|} \right) \cdot \hat{\mathbf{z}}) dv' \\
 &= \left(\frac{\sigma_0}{4\pi\epsilon_0} \iiint \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dv' \right) \cdot \hat{\mathbf{z}} \tag{1}
 \end{aligned}$$

From the usual expression for the electric field of a uniform charge distribution of density ρ we can observe that the field multiplying $\hat{\mathbf{z}}$ above is $\sigma_0 \mathbf{E}_1$. The field \mathbf{E} of the surface charge can be deduced from the potential $V(\mathbf{r})$.

$$\begin{aligned}
 \mathbf{E} &= -\bar{\nabla} V = -\bar{\nabla} (\sigma_0 \mathbf{E}_1 \cdot \hat{\mathbf{z}}) = -\sigma_0 \bar{\nabla} (\mathbf{E}_1 \cdot \hat{\mathbf{z}}) = -\sigma_0 (\hat{\mathbf{z}} \times (\bar{\nabla} \times \mathbf{E}_1) + (\hat{\mathbf{z}} \cdot \bar{\nabla}) \mathbf{E}_1) \\
 &= -\sigma_0 \frac{\partial \mathbf{E}_1}{\partial z}
 \end{aligned}$$

If \mathbf{E}_1 can be found out simply, for example when the enclosed region has a symmetrical shape, then this theorem makes a direct evaluation of \mathbf{E} unnecessary. For example, if the surface be spherical, then we know from Gauss's law that \mathbf{E}_1 is $(\mathbf{r}-\mathbf{r}')/3\epsilon_0$ at points inside the sphere, where \mathbf{r}' is the radius vector of the centre of the sphere. Since \mathbf{E}_1 varies such that $\partial\mathbf{E}_1 / \partial z$ happens to be a constant ($\hat{\mathbf{z}} / 3\epsilon_0$), we see that the field of the surface charge is a constant ($-\sigma_0\hat{\mathbf{z}} / 3\epsilon_0$) inside the sphere. This is the method of solution of Griffiths about which we had discussed in the abstract.

III. PHYSICAL RELEVANCE OF THE THEOREM

The theorem established in the last section has roots in an intuitive statement, and we shall call it as Griffith's normal charge theorem. It will be the license to a particular heuristic way of solution of some problems in physics³.

Let's get back to the situation in which our theorem is placed. S is a closed surface enclosing a region \mathbf{R} of any arbitrary shape. We now take an infinitesimally small distance Δz and make another region \mathbf{R}' by translating \mathbf{R} in negative z direction by a shift of Δz . Fig. 1 shows the situation.

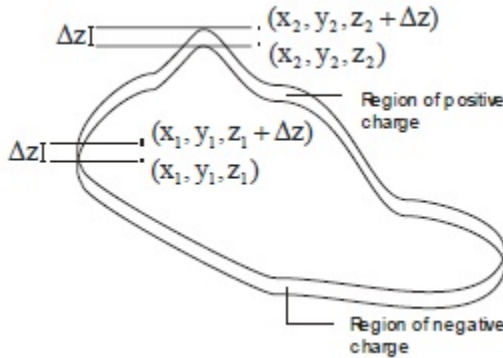


Fig. 1. Cross section of two identical regions with their geometrical centres (not in the figure) separated by the vector $\Delta z\hat{\mathbf{z}}$. The upper region carries a uniform positive charge, while the lower region carries a uniform negative charge of an equal magnitude. Two pairs of points separated by the vector $\Delta z\hat{\mathbf{z}}$ are depicted, to illustrate that a point $(x, y, z + \Delta z)$ is to the upper region what a point (x, y, z) is to the lower region.

Now we fill \mathbf{R} by a very large charge density $\sigma_0 / \Delta z$. We fill \mathbf{R}' oppositely by a charge density $-\sigma_0 / \Delta z$. The combined system is neutral except for some places near the periphery of the regions \mathbf{R} and \mathbf{R}' .

Now the field at any point (x, y, z) due to the charge in \mathbf{R} is $(\sigma_0/\Delta z)\mathbf{E}_1(x, y, z)$, where \mathbf{E}_1 is the field per unit charge density due to a uniform charge distribution in \mathbf{R} . It is evident as equality from Fig. 1 that the value of field per unit charge density at a general point (x, y, z) due to a uniform charge distribution in region \mathbf{R}' is the same as the

value of field per unit charge density at the point $(x, y, z + \Delta z)$ due to a uniform charge distribution in region \mathbf{R} . This is the crux of this section. Hence, the field at any point (x, y, z) due to the opposite charge in \mathbf{R}' is $-(\sigma_0 / \Delta z)\mathbf{E}_1(x, y, z + \Delta z)$. So the net field of the combined system is $-\sigma_0(\mathbf{E}_1(x, y, z + \Delta z) - \mathbf{E}_1(x, y, z))/\Delta z$, i.e. $-\sigma_0 \partial \mathbf{E}_1 / \partial z$.

But this is also the field due to the normal charge distribution $\sigma = \sigma_0 \cos \theta$ on S . Since the two fields are the same at all points, the two charge configurations must be the same. This is the conclusion: the normal charge theorem gives the physical insight into the working of this kind of a method of small relative shift in the evaluation of the field of a normal charge distribution over a sphere or any closed surface in general⁴.

IV. AN ALTERNATE FORM OF GRIFFITH'S THEOREM

Suppose ρ and \mathbf{d} are two constants having the dimensions of charge density and length respectively. A surface charge distribution of $\rho \mathbf{d} \cdot \hat{\mathbf{n}}$ over a closed surface will be a normal charge distribution. It follows from equation (1) that the potential V of the surface charge equals $\mathbf{E} \cdot \mathbf{d}$ where \mathbf{E} is the electric field of a uniform region charge ρ within the region enclosed by the surface.

This statement makes an alternate form of Griffith's theorem⁵. We can note that \mathbf{E} is independent of \mathbf{d} . Thus, for a given ρ we can chose any \mathbf{d} and we would have $\mathbf{E} \cdot \mathbf{d}$ as the corresponding potential $V(\mathbf{r})$. But the potential $V(\mathbf{r})$ is

$$\frac{1}{4\pi\epsilon_0} \iiint \frac{\rho \mathbf{d} \cdot \hat{\mathbf{n}}'}{|\mathbf{r} - \mathbf{r}'|} da' = \left(\frac{\rho}{4\pi\epsilon_0} \iiint \frac{da'}{|\mathbf{r} - \mathbf{r}'|} \right) \cdot \mathbf{d}$$

Writing $\mathbf{E} \cdot \mathbf{d} = V$ with V in this final form, we readily infer that

$$\mathbf{E} = \frac{\rho}{4\pi\epsilon_0} \iiint \frac{d\mathbf{a}'}{|\mathbf{r} - \mathbf{r}'|} \quad (2)$$

Since the equality holds for all \mathbf{d} . Here we have a statement for uniform region charges.

V. CONCLUDING REMARKS

Griffith's normal charge theorem makes some statements in electrostatics evident. For instance, the electric field at any point on the axis of z due to a volume charge of uniform density unity spilled in the cuboidal region described by the three inequalities $|x| \leq l/2$, $|y| \leq b/2$ and $|z| \leq h/2$ is numerically equal to the potential at that point due to a pair of surface charges of uniform densities plus unity and minus unity smeared respectively over the two rectangles, placed in the two different planes

$z = h/2$ and $z = -h/2$, and bounded by the same pair of inequalities $|x| = l/2$ and $|y| = b/2$. In addition, it can also help us establish mathematical statements. We'll demonstrate this with a short mathematical digression. Here we introduce ourselves to a new mathematical theorem on the intersection of closed surfaces.

Consider two closed surfaces S_1 and S_2 . These two surfaces may or may not intersect. It is intuitively evident that the integrals $\iint_{S_1} \left(\iint_{S_2} \frac{da_2}{|r_2 - r_1|} \right) \cdot da_1$ and

$\iint_{S_2} \left(\iint_{S_1} \frac{da_1}{|r_1 - r_2|} \right) \cdot da_2$ will both equal a common value which we denote as $\int \frac{da_1 \cdot da_2}{|r_1 - r_2|}$ for simplicity. Some can prove this proposition and then consider this nota-

tion to be defined – while others can naturally talk of it as a sum of a large number of products. We shall learn here that this integral is necessarily non-negative.

Let's now call the two closed surfaces S and S' . We denote by R the region enclosed by S and by R' the region enclosed by S' . Next we imagine R' to be filled uniformly by a region charge of density ρ , and evaluate the flux of the electric field of this charge distribution over the surface S . The situation is shown in Fig.2.

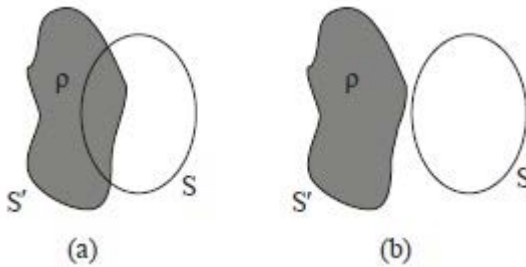


Fig. 2. Examples of two closed surfaces S and S' . The region enclosed by S' carries a uniform charge density ρ . In (a) the surfaces intersect and there is a non-zero volume of the common region enclosed by both S and S' . In (b) the common region is zero.

By the statement of Gauss's theorem we have the knowledge that the flux amounts to $\rho v_{enc} / \epsilon_0$, where v_{enc} is the volume of the region enclosed by both S and S' , i.e. the common region $R \cap R'$. Now, invoking the result of equation (2), we express the flux as

$$\iint_S \left(\frac{\rho}{4\pi\epsilon_0} \iint_{S'} \frac{da'}{|r - r'|} \right) \cdot da$$

As this equals $\rho V_{enc} / \epsilon_0$, we have

$$\iint_S \left(\iint_{S'} \frac{d\mathbf{a}'}{|\mathbf{r} - \mathbf{r}'|} \right) \cdot d\mathbf{a} = 4\pi V_{enc} \quad (3)$$

Switching the process, i.e. filling \mathbf{R} by ρ and considering flux over \mathbf{S}' , we get

$$\iint_{S'} \left(\iint_S \frac{d\mathbf{a}}{|\mathbf{r} - \mathbf{r}'|} \right) \cdot d\mathbf{a}' = 4\pi V_{enc} \quad (4)$$

Comparing equations (3) and (4), we see that

$$\iint_S \left(\iint_{S'} \frac{d\mathbf{a}'}{|\mathbf{r} - \mathbf{r}'|} \right) \cdot d\mathbf{a} = \iint_{S'} \left(\iint_S \frac{d\mathbf{a}}{|\mathbf{r} - \mathbf{r}'|} \right) \cdot d\mathbf{a}' \quad (5)$$

We assume a convention of denoting this common value as

$$\int \frac{d\mathbf{a} \cdot d\mathbf{a}'}{|\mathbf{r} - \mathbf{r}'|} = 4\pi V_{enc} \quad (6)$$

Our result is that this integral equals 4π times the volume V_{enc} of the common region enclosed by both of the two closed surfaces of integration. Hence, it equals zero if the surfaces do not intersect. It is zero even if \mathbf{S} and \mathbf{S}' do not intrude, i.e. even if they touch each other over a finite or infinitesimal area. If it is non-zero, then it is necessarily positive, and $1/4\pi$ of its value equals the volume of the common region. We call this theorem as the common region theorem. It teaches us the important fact that the integral

$\int \frac{d\mathbf{a} \cdot d\mathbf{a}'}{|\mathbf{r} - \mathbf{r}'|}$ is essentially non-negative.

VI. APPENDIX

In course of our derivation of the common region theorem we used concepts from electrostatics. But the theorem is a purely mathematical statement and must have a mathematical proof. With the benefit of hindsight we can see that after carefully surveying our proof a short direct proof can be obtained. We must take note here that this proof deprives us from all insight and knowledge in electrostatics that we got when we spontaneously derived the theorem.

If $f(\mathbf{r}_1, \mathbf{r}_2)$ is any scalar function of two independent coordinates \mathbf{r}_1 and \mathbf{r}_2 , and \mathbf{S}_1 and \mathbf{S}_2 are any two closed surfaces enclosing two connected regions \mathbf{R}_1 and \mathbf{R}_2 respectively, then, according to a well-known corollary of the divergence theorem

$\oint_{S_1} f(\mathbf{r}_1, \mathbf{r}_2) d\mathbf{a}_1 = \iiint_{R_1} \bar{\nabla}_1 f(\mathbf{r}_1, \mathbf{r}_2) dV_1$. Taking f to be $\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}$ we have

$$\oint_{S_1} \frac{d\mathbf{a}_1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \iiint_{R_1} \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} dV_1. \text{ So,}$$

$$\begin{aligned} \oint_{S_2} \left(\oint_{S_1} \frac{d\mathbf{a}_1}{|\mathbf{r}_2 - \mathbf{r}_1|} \right) \cdot d\mathbf{a}_2 &= \iiint_{R_2} \bar{\nabla}_2 \cdot \left(\oint_{S_1} \frac{d\mathbf{a}_1}{|\mathbf{r}_2 - \mathbf{r}_1|} \right) dV_2 \\ &= \iiint_{R_2} \bar{\nabla}_2 \cdot \left(\iiint_{R_1} \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} dV_1 \right) dV_2 \\ &= \iiint_{R_2} \left(\iiint_{R_1} \bar{\nabla}_2 \cdot \left(\frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \right) dV_1 \right) dV_2 \\ &= \iiint_{R_2} \left(\iiint_{R_1} 4\pi \delta^3(\mathbf{r}_2 - \mathbf{r}_1) dV_1 \right) dV_2 \\ &= 4\pi V_{\text{enc}} \end{aligned}$$

As $\iiint_{R_1} 4\pi \delta^3(\mathbf{r}_2 - \mathbf{r}_1) dV_1$ is 4π if \mathbf{r}_2 is inside R_1 and zero if it's outside.

ACKNOWLEDGEMENTS

I thank Karthik Chikmagalur, one senior of mine at IITG who has completed his graduation now, for bringing to me the proof in the appendix while he was studying here. I thank my father, Mr. Shyam Rudra Pathak, for pointing out some nice modifications in the first form of the manuscript.

FOOTNOTES AND REFERENCES

- [1] David J. Griffiths, *Introduction to Electrodynamics* (Prentice Hall, Upper Saddle River, NJ, 1999), 3rd ed., pp. 168, 169 and 172, example 4.2 and example 4.3.
- [2] David J. Griffiths, "The field of a uniformly polarized object," *Am. J. Phys.* **60**(2), 187 (1992).
- [3] I.E.Irodov, *Problems in General Physics* (Mir Publishers Moscow, 1981), p. 107, Problem 3.17.
- [4] For a general surface, as discussed in section II, theta stands for the angle that the outward normal forms with the positive direction of z; for the theorem it is simply coincidence that theta happens to be the polar angle in case of a sphere.
- [5] In fact, this is the form in which the theorem was actually mentioned in Griffith's work.