A Generalization Of Gauss's Theorem In Electrostatics

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Abstract— **Gauss's theorem of electrostatics states that the flux of the electrostatic field over a closed surface equals** $\oiint_S \mathbf{E} \cdot \boldsymbol{da} = Q_{enc}/\varepsilon_0$ **, where** Q_{enc} **equals the net charge enclosed by S. In the derivation it is assumed that no charge lies on the surface in question. Consider the problem of evaluation of the electrostatic field due to a uniformly charged spherical surface on the surface itself. The situation exhibits symmetry but we can't apply the Gauss's theorem, and we have to resort to other methods like direct integration. In this paper we prove a generalization of Gauss's theorem which allows charges to lie on the surface of integration. For the majority of cases the statement of our generalized Gauss's theorem can be assumed to be this: the flux of electrostatic field over a closed surface equals**

$$\oiint_S \mathbf{E} \cdot \boldsymbol{da} = \frac{Q_{enc}}{\varepsilon_0} + \frac{1}{2}\frac{Q_{con}}{\varepsilon_0}$$

, where Q_{enc} **is the net charge enclosed by S and** Q_{con} **is the net charge contained by S. Applying this theorem to the uniformly charged spherical surface we find at once that the field equals exactly half of the field which would have existed if the charge lied completely inside the surface in a spherically symmetric manner. Using this generalization of Gauss's theorem we present a generalized electrostatic boundary condition, which we then use to solve the famous conducting plane image problem without using the method of images.**

I. INTRODUCTION

One of the most important theorems of electrostatics is the Gauss's theorem. The well-known theorem states $\oiint_S \mathbf{E} \cdot \boldsymbol{da} = Q_{enc}/\varepsilon_0$. It is explicitly stated sometimes[1], that the boundary of the region enclosed must not contain any point, line or surface charges. But what happens if the surface contains such charges in any idealized problem? To my knowledge, the generalization of Gauss's theorem has not been discussed anywhere till now, and I present it here. For the next few sections we discuss what missing terms appear in the gauss's theorem if we permit presence of charges on the surface of integration.

II. FLUX OF THE ELECTROSTATIC FIELD OF A POINT CHARGE

We prove in this section that the flux of the electrostatic field $\mathbf{E_i}$ of a charge q_i over a boundary S to a connected region R_{enc} is

$$\oiint_S \mathbf{E_i} \cdot da = \begin{cases} 0 & \text{if } q_i \text{ lies outside S} \\ \dfrac{q_i \Omega_i}{4\pi\varepsilon_0} & \text{if } q_i \text{ lies on S at a point where} \\ & \quad \text{inside solid angle is } \Omega_i \\ \dfrac{q_i}{\varepsilon_0} & \text{if } q_i \text{ is enclosed by S} \end{cases} \qquad (1)$$

This statement is axiomatic[2] but still I'll provide a satisfactory proof soon. Yet before doing that I'd like to expatiate a little. An analogy is often drawn between the actual situation and an imaginary situation with the point charge radiating photons at a constant rate in a spherically symmetric manner. Since the electrostatic field is given by a radial inverse square law equation, the total number of photons passing per unit time through a surface can be thought of to be the flux of the electrostatic field through that surface. Now if we consider a charge lying on the closed surface under consideration, the total number of photons passing per unit time through the closed surface should be obtained by multiplying the total number of photons emitted by the charge per unit time by $1/4\pi$ times the inside solid angle formed at the surface at the place where the point charge lies. Now let me show my proof for equation 1. If the reader is convinced by the above reasoning then the rest of this section can be skipped.

We denote by $\mathbf{r_i}$ the position of q_i. Let's first consider the case A when the charge lies outside the boundary S. In this case $\mathbf{E_i}$ is differentiable properly over an open connected region containing the surface S along with its enclosure. Application of divergence theorem will yield the result. Let's now consider the case B when $\mathbf{r_i}$ lies on S. Let the inside solid angle formed on S at the point $\mathbf{r_i}$ be Ω_i in measure. We chose an arbitrary non-zero radius δ sufficiently small, say less than a critical radius δ_c, such that the part S_δ of the spherical surface of radius δ centered at $\mathbf{r_i}$, which lies not outside the region R_{enc} enclosed by S, satisfies the following two conditions

i)   It divides R_{enc} into two parts $R_{1\delta}$ and $R_{2\delta}$ such that $R_{2\delta}$, the one not adjacent to $\mathbf{r_i}$, is enclosed by a closed surface $S_{2\delta}'$ which has the inner surfaces of S as its inner surfaces in case R_{enc} has cavities.

ii)   It has no missing patches, i.e. S_δ is bounded by a single closed curve P_δ and not by a group of closed curves.

Hence, the closed curve P_δ divides the outer surface of S into two parts, $S_{1\delta}$ and $S_{2\delta}$, where $S_{2\delta}$ and S_δ together with the inner surfaces of S form the closed surface $S_{2\delta}'$

which encloses $R_{2\delta}$. Also $S_{1\delta}$ and S_{δ} together form the closed surface $S_{1\delta}'$ which encloses the compact region $R_{1\delta}$. Now, we can write the flux of electrostatic field through S as

$$\oiint_S \mathbf{E_i} \cdot \boldsymbol{da} = \oiint_{S_{1\delta}'} \mathbf{E_i} \cdot \boldsymbol{da} + \oiint_{S_{2\delta}'} \mathbf{E_i} \cdot \boldsymbol{da}$$

Because when we add the flux

$$\Phi = \frac{q_i}{4\pi\varepsilon_0} \frac{\text{area of } S_{\delta}}{\delta^2}$$

Of $\mathbf{E_i}$ through S_{δ} to the second integral on the right of equality and subtract the same from the first one, we get identity. Now according to the result of case A the second integral must vanish. The flux of $\mathbf{E_i}$ through S can now be easily obtained

$$\oiint_S \mathbf{E_i} \cdot \boldsymbol{da} = \lim_{\delta \to 0} \oiint_S \mathbf{E_i} \cdot \boldsymbol{da} = \lim_{\delta \to 0} \oiint_{S_{1\delta}'} \mathbf{E_i} \cdot \boldsymbol{da} = \lim_{\delta \to 0} \Phi = \frac{q_i \Omega_i}{4\pi\varepsilon_0}$$

Finally we consider the case C when $\mathbf{r_i}$ lies enclosed by S. We consider any plane passing through $\mathbf{r_i}$ and denote by S_0 the part of the plane that is enclosed by S. We denote again by R_1 and R_2 the regions into which S_0 divides R_{enc}, and by S_1' and S_2' the closed surfaces that enclose these regions. So we have again

$$\oiint_S \mathbf{E_i} \cdot \boldsymbol{da} = \oiint_{S_1'} \mathbf{E_i} \cdot \boldsymbol{da} + \oiint_{S_2'} \mathbf{E_i} \cdot \boldsymbol{da}$$

Only this time we are already with an identity, for this time the flux Φ of $\mathbf{E_i}$ through S_0 is zero. Now according to the result of case B, both the terms on the right of equality are equal to $q_i/2\varepsilon_0$, so that the flux is

$$\oiint_S \mathbf{E_i} \cdot \boldsymbol{da} = \frac{q_i}{\varepsilon_0}$$

III.   THE GENERALIZED GAUSS'S THEOREM

The principal of superposition for electrostatic field allows us to insist that the flux of the electrostatic field over a closed surface S is $\oiint(\sum \mathbf{E_i}) \cdot \boldsymbol{da}$, where the summation is done over all charges in the charge configuration. The linearity of the operation of flux enables us to insist that the flux is $\sum(\oiint \mathbf{E_i} \cdot \boldsymbol{da})$. Since the flux $\oiint \mathbf{E_i} \cdot \boldsymbol{da}$ of any charge q_i enclosed by S is q_i/ε_0, the sum of the isolated fluxes of all charges enclosed by S is Q_{enc}/ε_0, where Q_{enc} is the net charge enclosed by S. Similarly we get that the

sum of the isolated fluxes of all charges lying outside S is zero, and that of all charges residing on the continuities of S is $Q_{con}/2\varepsilon_0$, where Q_{con} is the net charge residing on the continuities of S. By continuities of S, we mean a point on S where the principle curvatures of S vary smoothly so that the inside solid angle formed is equal to 2π. The sum of the isolated fluxes of the remaining charges (all of which lie at the discontinuities of S) is left as a summation

$$\Phi_d = \sum_{discon} \frac{\Omega_i q_i}{4\pi\varepsilon_o} \qquad (2)$$

For we can't say anything about the inner solid angles Ω_i without a particular knowledge of the geometry of the closed surface and the locations of the charges lying at the discontinuities. Hence, we have the generalized Gauss's theorem:

"The flux of the electrostatic field $\mathbf{E}$ over any closed surface S is

$$\oiint_S \mathbf{E} \cdot d\boldsymbol{a} = \frac{Q_{enc}}{\varepsilon_0} + \frac{1}{2}\frac{Q_{con}}{\varepsilon_0} + \Phi_d \qquad (3)$$

Where Q_{enc} equals the net charge enclosed by S, Q_{con} equals the net charge residing on the continuities of S and Φ_d, as described by (2), equals the flux of the electrostatic field of the charges lying at the discontinuities of S"

IV. VARIOUS FORMS OF THE GENERALIZED GAUSS'S THEOREM

We in this section consider those cases in which the generalized Gauss's theorem takes a beautiful form which we shall call as 'the simplest form of the theorem'. We begin by confining ourselves to an electrostatic field caused by a configuration of charges not containing any point or line charges. In this special case Q_{con} of equation (3) can also be interpreted as the net charge contained by S, for the net charge lying on the discontinuities of S is zero anyway, because S now contains no point or line charges- which were the only varieties which could accumulate to a finite amount by assembling only at discontinuities. Also Φ_d vanishes in case of an electrostatic field caused by such a configuration. This needs some explanation. Let's imagine a different source charge configuration- the one in which each charge is replaced by a corresponding positive one of an equal magnitude. The fact that the net charge lying on the discontinuities of S is zero whenever the charge configuration does not contain any point or line charges implies that for the original charge configuration we'll have $\sum_{discon}|q_i| = 0$.

Now since $-|q_i| \le q_i \le |q_i|$, we have

$$-\sum_{discon}\frac{|q_i|\Omega_i}{4\pi\varepsilon_0} \le \sum_{discon}\frac{q_i\Omega_i}{4\pi\varepsilon_0} \le \sum_{discon}\frac{|q_i|\Omega_i}{4\pi\varepsilon_0}$$

The central term in the inequalities is Φ_d. So,

$$\left| \Phi_d \right| \leq \left| \sum_{discon} \frac{\left| q_i \right| \Omega_i}{4\pi\varepsilon_0} \right| = \sum_{discon} \frac{\left| q_i \right| \Omega_i}{4\pi\varepsilon_0}$$

And since $0 \leq \Omega_i \leq 4\pi$, $0 \leq \left| q_i \right| \Omega_i \leq \left| q_i \right| 4\pi$. Hence,

$$0 \leq \sum_{discon} \frac{\left| q_i \right| \Omega_i}{4\pi} \leq \sum_{discon} \left| q_i \right|$$

As the rightmost term in the inequalities is zero, as argued earlier, we have the middle term equal to zero. From here we conclude that Φ_d vanishes. Therefore if the source charge configuration is free from point and line charges, then

"The flux of electrostatic field over any closed surface is

$$\oiint_S \mathbf{E} \cdot \boldsymbol{da} = \frac{Q_{enc}}{\varepsilon_0} + \frac{1}{2}\frac{Q_{con}}{\varepsilon_0} \qquad (4)$$

Where Q_{enc} is the net charge enclosed by S and Q_{con} is the net charge contained by S."

Superposing the fluxes we get as a corollary that for any kind of charge configuration

$$\oiint_S \mathbf{E} \cdot \boldsymbol{da} = \frac{Q_{enc}}{\varepsilon_0} + \frac{1}{2}\frac{Q_{sur}}{\varepsilon_0} + \Phi_{pl} \qquad (5)$$

Where Q_{enc} is the net charge enclosed by S, Q_{sur} is the net surface charge residing on S and Φ_{pl} is the flux of the electrostatic field of the point and line charges residing on S.

Till now (4) was referred to as applicable only in case of charge configurations that didn't contain point and line charges. From (5) it can be seen that (4) holds 'whenever no point or line charge lies anywhere on the discontinuities of S', for in that case we see (by referring to equation (3)) that $\Phi_{pl} = Q_p/2\varepsilon_0 + Q_l/2\varepsilon_0$, where Q_p equals the net point charge and Q_l equals the net line charge residing on S. And then, since the recent most restriction holds if S has no discontinuity at all, (4) holds 'whenever S is throughout continuous'. We shall call equation (4) as 'the simplest form of the generalized Gauss's theorem' and see that it is almost always applicable.

V. Concluding Remarks

In the introduction I mentioned that for the majority of cases the statement of our generalized Gauss's theorem can be assumed to be this: the flux of electrostatic field over a closed surface equals 1/vacuum permittivity times the sum of the net charge enclosed by S and half the value of net charge contained by S. Let me now enlist the cases that I claim to be in majority

- The surface of integration doesn't contain point or line charges at any of the corners or edges.

- The surface of integration doesn't have any edge or corner, i.e. it is throughout continuous.
- The source charge configuration consists of only volume charges and surface charges.

VI. THE GENERALIZED ELECTROSTATIC BOUNDARY CONDITION

Let S be a surface on which at a point $\mathbf{r}$ and "near the point $\mathbf{r}$, on one side of the surface (say side 1)", only volume and surface charges lie. Let's assume that this point isn't a discontinuity of the surface and denote by $\hat{\mathbf{n}}_1$ the unit normal to the surface pointing towards side 1. Let's denote the charge density on the surface S at the point $\mathbf{r}$ by $\sigma(\mathbf{r})$. We're here seeking a relationship between the limit $\mathbf{E}_1$ of the electrostatic field as we approach $\mathbf{r}$ from side 1, and the value $\mathbf{E}$ of the electrostatic field at the point $\mathbf{r}$ on the surface. Here I'm considering the electrostatic field to be of the form of a mathematical function which is well defined at a point of interest where the function value differs from the right hand limit. We draw as a Gaussian surface a closed surface of the form of a very thin geometry-box with the plane base of an extremely small area A lying on the surface and the remaining part extending in the direction of side 1 to an infinitesimal distance ε from the surface. The application of the generalized Gauss's theorem, which applies here in its simplest form, gives us

$$\oiint_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{enc}}{\varepsilon_0} + \frac{1}{2}\frac{Q_{con}}{\varepsilon_0} = \frac{Q_{enc}}{\varepsilon_0} + \frac{\sigma A}{2\varepsilon_0}$$

Now, in the limit at the thickness ε goes to zero, Q_{enc} goes to zero. Also in this limit the flux $\oiint_S \mathbf{E} \cdot d\mathbf{a}$ equals $\mathbf{E}_1 \cdot A\hat{\mathbf{n}}_1 - \mathbf{E} \cdot A\hat{\mathbf{n}}_1$, as in this limit, the sides of the box contribute nothing to the flux. So we get $(\mathbf{E}_1 - \mathbf{E}) \cdot \hat{\mathbf{n}}_1 = \sigma(\mathbf{r})/2\varepsilon_0$. As the tangential component of electrostatic field is always continuous at a surface charge, our sought for relationship between $\mathbf{E}_1$ and $\mathbf{E}$ becomes

$$\mathbf{E}_1(\mathbf{r}) - \mathbf{E}(\mathbf{r}) = \frac{\sigma(\mathbf{r})}{2\varepsilon_0}\hat{\mathbf{n}}_1 \tag{6}$$

We shall call this equation as the absolute electrostatic boundary condition for it can be used to arrive at the usual boundary equation in this manner:

$$\mathbf{E}_1 - \mathbf{E} = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{n}}_1 \,; \mathbf{E}_2 - \mathbf{E} = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{n}}_2 = -\frac{\sigma}{2\varepsilon_0}\hat{\mathbf{n}}_1 \,; \mathbf{E}_1 - \mathbf{E}_2 = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}_1$$

Question: A point charge is placed at a distance d from an infinite conducting plane; what is the charge density on the plane at a distance r from the foot of the perpendicular to the plane from the point charge?

This is the simplest problem for which the method of images is invoked. I propose to use the absolute boundary condition to solve this problem without using the method of

images. We select $\hat{\mathbf{n}}_1$ to point inside the conductor so that $\mathbf{E_1}$ vanishes. Generalized boundary condition then gives $-\mathbf{E} \cdot \hat{\mathbf{n}}_1 = \sigma(\mathbf{r})/2\varepsilon_0$. Now, from the principal of superposition $\mathbf{E}(\mathbf{r}) = \mathbf{E_q}(\mathbf{r}) + \mathbf{E_S}(\mathbf{r})$, where $\mathbf{E_q}$ equals the electric field of the point charge q and $\mathbf{E_S}$ equals the electric field of the surface charges on the conducting plane. It follows straight from the Coulomb's law that for any point $\mathbf{r}$ on the conducting plane

$$\mathbf{E_q}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0} \frac{d(\hat{\mathbf{n}}_1) + \mathbf{r}}{(d^2 + r^2)^{3/2}}$$

So, we have

$$-\mathbf{E_q}(\mathbf{r}) \cdot \hat{\mathbf{n}}_1 = \frac{q}{4\pi\varepsilon_0} \frac{-d}{(d^2 + r^2)^{3/2}}$$

Because $\mathbf{r}$ is orthogonal to $\hat{\mathbf{n}}_1$. Utilizing this in the equation $-\mathbf{E} \cdot \hat{\mathbf{n}}_1 = \sigma(\mathbf{r})/2\varepsilon_0$ we get

$$\frac{\sigma(\mathbf{r})}{2\varepsilon_0} = -\frac{qd}{4\pi\varepsilon_0(d^2 + r^2)^{3/2}} - \mathbf{E_S}(\mathbf{r}) \cdot \hat{\mathbf{n}}_1$$

Now, since the electrostatic field of any charge points radially away from the charge, we must have $\mathbf{E_S}$ orthogonal to $\hat{\mathbf{n}}_1$. This leads us straight to the solution of the conducting plane image problem:

$$\sigma(\mathbf{r}) = \frac{-1}{2\pi} \frac{qd}{(r^2 + d^2)^{3/2}} \qquad (7)$$

This is a well-known equation, and at present it is held by the physics community that it can be derived only using the method of images.

VII. Appendix

In our solution above, we assumed the electrostatic field to behave as an idealized mathematical field. In fact, the field on the surface is discontinuous and it has no well-defined value on the surface. The ambiguity can be removed by talking about the electrostatic force per unit area on the surface charge instead of the electrostatic field.

Let the charge q be at $d\hat{\mathbf{k}}$ and the upper surface of the conducting plane be the X-Y plane. On the plane $r \equiv (x^2 + y^2)^{1/2}$ is the distance from the origin. At any point in space the total field is due in part to q and in part to the surface charges induced on the plane: $\mathbf{E} = \mathbf{E_q} + \mathbf{E_S}$.

The electrostatic force per unit area on the surface is $\mathbf{f} = \sigma^2(\mathbf{r})\hat{\mathbf{k}}/2\varepsilon_0$. Now, $\mathbf{f}(\mathbf{r}) = \sigma(\mathbf{r})\mathbf{E_q}(\mathbf{r}) + \mathbf{f_S}(\mathbf{r})$, where $\mathbf{f_S}(\mathbf{r})$ is the electrostatic force per unit area on the X-Y plane due to the charges on the plane. By symmetry, it can have no Z-component.

Equating the above two values of $\mathbf{f}$ and taking the dot product with $\hat{\mathbf{k}}$ we get $\sigma(\mathbf{r})/2\varepsilon_0 = \mathbf{E}_q(\mathbf{r}) \cdot \hat{\mathbf{k}} = -qd/4\pi\varepsilon_0(\mathbf{r}^2 + \mathbf{d}^2)^{3/2}$. So, $\sigma(\mathbf{r}) = -qd/2\pi(\mathbf{r}^2 + \mathbf{d}^2)^{3/2}$.

Let me present another solution without using the method of images or any theorem presented in this paper. At a general point P on the plane whose distance from the origin is r, we have $(E_q)_z = -qd/4\pi\varepsilon_0(\mathbf{r}^2 + \mathbf{d}^2)^{3/2}$, and we know $(E_q)_z$ is continuous. On the other hand $(E_S)_z$ is discontinuous in the amount σ/ε_0, and by symmetry is the same above and below in magnitude and its direction on both sides is either towards the plane or away from the plane. Thus immediately below the plane $(E_S)_z = -\sigma/2\varepsilon_0$. But below the plane, i.e. inside the conductor, the total field is zero. So, $(E_q)_z + (E_S)_z = 0$. Hence, $\sigma(\mathbf{r}) = -qd/2\pi(\mathbf{r}^2 + \mathbf{d}^2)^{3/2}$.

Let me present yet another solution. I learnt it from Prof. J.D.Jackson in the reply to a mail conveying my solution.

Consider any point P on the plane. On the conducting plane, the electrostatic field caused by the point charge and the distribution of surface charges must be normal to the plane. Otherwise, the charge, free to move, will readjust itself. Now, since the tangential component of electrostatic field is continuous at a surface charge, the field is normal to the plane both above and below P. If the field just above P due to the surface charges is $-E_S(\cos\alpha\hat{\mathbf{k}} + \sin\alpha\hat{\mathbf{r}})$ (where $\hat{\mathbf{r}}$ is a unit vector in the plane pointing from the origin O to the point P), then by symmetry, just below P it is $E_S(\cos\alpha\hat{\mathbf{k}} - \sin\alpha\hat{\mathbf{r}})$. Since, just above P the net field is normal, we have $E_S\sin\alpha = E_q\sin\theta$. And just below P the net field is zero, so $E_S\cos\alpha = E_q\cos\theta$. Hence $E_S = E_q$ and $\alpha = \theta$. Thus, we see that the net field just above P is $-2E_q\cos\theta\,\hat{\mathbf{k}}$. Application of Gauss's theorem to a pill-box at P that spans the surface (with zero contribution from the side of the box within the conductor) gives $\sigma(\mathbf{r}) = -2\varepsilon_0 E_q\cos\theta = -qd/2\pi(\mathbf{r}^2 + \mathbf{d}^2)^{3/2}$.

By arguing without using an image charge we, in our solution, showed that at any point on the conducting plane the field due to induced surface charges on the plane is constructed by first reflecting q's field in the conducting plane and then reversing its direction. This is exactly the field of an image charge –q placed at $-d\hat{\mathbf{k}}$, but we notice that after the fact.

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point nearest to one of the foci. So, in a sense, this work is his in a way which makes him deny this claim of mine himself.

The second proof in the appendix is an altered version of a proof which a referee suggested when I conveyed a paper titled 'a solution of the conducting plane image problem without using the method of images' to American Journal of Physics. The paper was assigned the manuscript # 22928, and the referees were interested in the solution to the image problem but not in the generalization. When I responded after a month being busy with my exams, I was told that the journal had received a couple months before my submission a paper on the same topic as that of mine. Reading this I published my work online immediately at http://www.scribd.com/doc/23461034/A-Solution-of-the-Conducting-Plane-Image-Problem-Without-Using-the-Method-of-Images

The last two paragraphs in the appendix come into this paper (after several modifications) from an e-mail sent to me by Prof. J.D.Jackson. I thank him for extracting several moments from his precious time, and encouraging me in my pursuit of knowledge in physics.

FOOTNOTES AND REFERENCES

[1] E.M.Purcell, Electricity and Magnetism, 1st edition, (McGraw Hill, 1965), p.22.

[2] In many texts it is pointed out that the flux of the electrostatic field of a point charge through any infinitesimal surface (with the normal to the surface pointing away from the charge) is $qd\Omega / 4\pi\varepsilon_0$, where $d\Omega$ is the solid angle subtended by the infinitesimal surface at the position of the charge. E.g. See J.D.Jackson, Classical Electrodynamics, 3rd edition, (John Wiley & Sons, 1999), pp.27-28. Once this result is known, one can say at once that the flux of the field of a single point charge through a closed surface is given by eq. (1).