Exact and approximate expressions for the force between a charged conducting sphere and infinite grounded plane

Shubho Banerjee, Stefan J. McCarty, and Evan F. Nelsen
Dept. of Physics
Rhodes College
phone: (1) 901-843-3585
e-mail: banerjees@rhodes.edu

***Abstract*—The geometry of a charged conducting sphere interacting with an infinite grounded plane can be used to model applications where an object interacts electrostatically with another object that is much larger than its own size. We provide an exact closed-form expression for the electrostatic force for this geometry using the q-analog of the digamma function that avoids the use of the infinite series solutions. Furthermore, we provide a simpler algebraic expression for the electrostatic force that gives accurate results to within 0.4% of the exact result at all distances. Our results can be useful for computer algorithms modeling the dynamics of particles where a quick calculation of the electrostatic force is needed.**

I. Introduction

The interaction of a charged conducting sphere with a grounded infinite conducting plane was first analyzed in the late 19th century [1]. The solution of this problem is of practical interest since this geometry can be used as a simple model in many applications such as atomic force microscopy [2], dust adhesion [3], xerographic printing [4] etc.

The geometry of the problem is shown in Fig. 1. A sphere of radius R is held at a fixed voltage V such that the center of the sphere is at distance r from the plane. Using the method of images [5], the interaction of the sphere with the plane can be calculated through a series of equivalent interactions between image charges within sphere A with image charges of opposite polarity within sphere A' (see Fig. 1).

The capacitance, C, of the sphere-plane system is calculated by summing up the image charges within sphere A and dividing by its voltage V to give the classical result [5]

$$c = \frac{C}{4\pi\varepsilon R} = \sinh\alpha \sum_{n=1}^{\infty} [\sinh(n\alpha)]^{-1}, \tag{1}$$

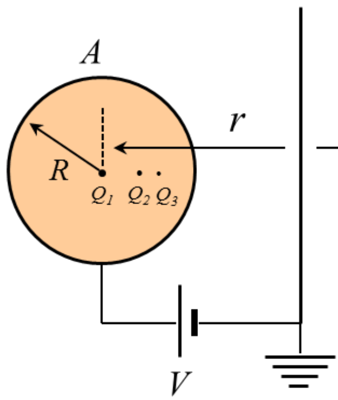

Fig.1. Basic geometry of the problem is shown. The interaction of the sphere with the plane can be summed up as a series of interactions of the image charges Q_1, Q_2, … located inside A with images of opposite polarity located within A', the image sphere of A.

where $\alpha = \cosh^{-1}(r/R)$, c is the dimensionless form of C, and ε is the permittivity of the medium. The infinite series solution in Equation 2 was evaluated numerically by Pisler and Adhikari [6] accurately to better than one part in 10^6. Increasing number of terms are needed to achieve the desired accuracy as the surface of sphere approaches close to the plane ($r/R \approx 1$).

To avoid the slow convergence of the infinite series solutions Hudlet et al. [2] derived an accurate approximation for the capacitance. This approximation, accurate to within 2%, was later expressed by Crowley [7] in elegant dimensionless form as

$$\tilde{c} = 1 + \tfrac{1}{2}\log(1 + \tfrac{1}{\xi}), \tag{2}$$

where $\xi = (r - R)/R$ is the relative sphere-plane separation. Using this approximation the force (at constant voltage) on the sphere can be calculated to within 5%.

In this paper we provide an exact closed-form expression for the capacitance. Using these exact results we improve upon accuracy of the existing approximations for the capacitance and the force by more than one order of magnitude. We follow the same dimensionless notation introduced by Crowley in Reference [7].

III. EXACT CLOSED-FORM CAPACITANCE AND FORCE

To calculate the capacitance of the sphere-plane system in closed form we define

$$x \equiv \frac{r - \sqrt{r^2 - R^2}}{R} = 1 + \xi - \sqrt{\xi(2 + \xi)} \ . \tag{3}$$

Using x, the capacitance series in Equation 1 can be re-expressed as

$$c = \frac{1-x^2}{x}\left(\sum_{n=1}^{\infty}\frac{x^n}{1-x^n} - \sum_{n=1}^{\infty}\frac{x^{2n}}{1-x^{2n}}\right). \tag{4}$$

The Lambert series in the parentheses can be summed up [8] to give

$$c = \left(\frac{1-x^2}{x}\right)\left(\frac{2\psi_x(1) - \psi_{x^2}(1) - 2\tanh^{-1} x}{2\log x}\right), \tag{5}$$

where $\psi_q(z)$ is the q-analog of the digamma function of z [9]. This new result exactly reproduces the Pisler and Adhikari's numerical results [6].

The force at constant voltage, F_V, can be calculated as the derivative of the electrostatic energy, $1/2\, CV^2$, w.r.t. distance and expressed in dimensionless form as

$$f_V = \frac{4x}{\log x}\left(\frac{d\psi_x(1)}{dx} - x\frac{d\psi_{x^2}(1)}{dx^2} - \frac{1}{1-x^2}\right) - \frac{4xc}{1-x^2}\left(\frac{2x}{1-x^2} + \frac{1}{x} + \frac{1}{x\log x}\right) \tag{6}$$

where c is the capacitance expression in Equation 5 and $f_V = F_V/\pi\varepsilon V^2$. The force on the sphere at constant charge, $F_Q = (Q^2/16\pi\varepsilon R^2) f_Q$, where $f_Q = f_V/c^2$ as derived in [7].

IV. APPROXIMATE CAPACITANCE AND FORCE

While the expressions in Equations 5 and 6 give the exact result for this problem, they involve the special q-digamma function. To our knowledge, *Mathematica* is the only software that carries this function. Therefore, there is a need for simpler expressions.

By analyzing Equation 5, we improve upon the approximation in Equation 2 with

$$\tilde{c}_2 = 1 + \frac{1}{2}\log\frac{1+x^2}{(1-x)^2} + k\left(\frac{2x-x^2}{2-2x-x^2}\right)^{2.15}, \tag{7}$$

where the coefficient $k = \gamma + ½ \log 2 - 1 = -0.0762\ldots$ The additional k-term improves the approximation accuracy by more than one order of magnitude from 2% to about 0.05%.

Using the better approximate capacitance in Equation 7 we calculate the force on the sphere

$$\tilde{f}_{V2} = \frac{4x^2}{(1+x^2)(1-x)^2} - 2.62\frac{x^2(2x-x^2)^{1.15}}{(1+x)(2-2x+x^2)^{3.15}} \quad . \tag{8}$$

The maximum error in Equation 8 is only 0.4%. The errors in approximations in Equations 7 and 8 are plotted in Fig 2 below along with the original approximations in References [2, 7]. The force at constant charge can be calculated using Equations 7 and 8 in the same way as outlined in Section III. Our approximations for the constant charge case are accurate to within 0.4%.

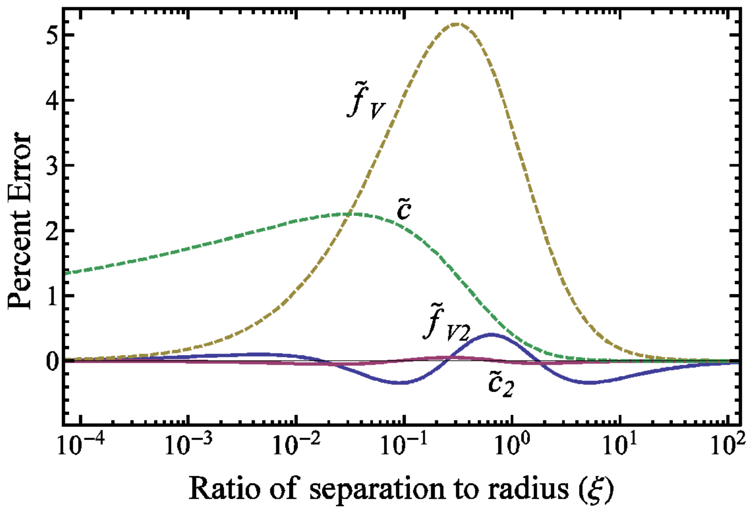

Fig. 3. The percent error in our approximate capacitance and force expressions are plotted versus the relative separation. Also plotted (dashed lines) are the original approximations [2, 7]. Our corrections in Equations 7 and 8 improve the accuracy of the approximations by one order of magnitude.

References

[1] J. C. Maxwell, *A Treatise on Electricity and Magnetism*, 3rd ed. (Dover, 1954).

[2] S. Hudlet et al., "Evaluation of the capacitive force between an atomic force microscopy tip and a metallic surface," *Eur. Phys. J. B*, vol. 2, pp. 5-10, 1998.

[3] A. D. Zimon, *Adhesion of Dust and Powder*, (Plenum, New York, 1982).

[4] L. B. Schein, *Electrophotography and Development Physics*, (Laplacian, Morgan Hill, 1996).

[5] W. R. Smythe, *Static and Dynamic Electricity* (McGraw-Hill, New York, 1950).

[6] E. Pisler and T. Adhikari, "Numerical Calculation of mutual capacitance between two equal metal spheres," *Physica Scripta*, vol. 2, pp. 81-83, 1970.

[7] J. M. Crowley, "Simple expressions for force and capacitance for a conductive sphere near a conductive wall," *Proc. ESA Annu. Meet. Electrostatics*, Paper D1, 2008.

[8] J. M. Borwein and P. B. Borwein, *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity*, (Wiley, New York, 1987), pp. 91–101.

[9] C. Krattenthaler, H.M. Srivastava, "Summations for basic hypergeometric series involving a q-analogue of the digamma Function", *Comput. Math. Applic.* 32 (1996) 73-91.