The Electric Field Produced by a Non-Conducting Web of Finite Size

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Abstract

A non-conducting material is referred to as an insulator because the electrical relaxation time of the material is very large. As a result, any charges placed on its surface remain on its surface for a very long time. In a manufacturing process when a non-conducting web is transported past rollers the continual contact and separation between the web and each roller of the transport system can result in contact charging of the web surface. This contact charging, also known as triboelectrification, can produce a web with sufficient surface charge to create -under certain conditions- an electrical discharge which in some situations might give rise to an electrical shock or cause a fire or explosion. To prevent these problems, much effort is spent on reducing the surface charge on non-conducting webs during their transport. The steps taken to reduce surface charge are first to measure it by measuring the field it produces with an electrostatic fieldmeter, then to apply charge of the opposite polarity to reduce or neutralize it, and then to measure it again to insure it has been sufficiently reduced. If the web is treated as a uniformly-charged flat-plane or disk of infinite extent, then Gauss's law shows the electric field is everywhere constant, independent of position and proportional to the surface charge density on the web. However, in practice, the fieldmeter does not give the same reading everywhere. In this paper Coulomb's law is used to calculate the electric field at the center above both a uniformly-charged flat-disk of radius R and a uniformly-charged free-span of web of length L and width W. Webs of large L/W aspect ratio such as filaments and fibers are also included. The affect of field variation with distance from the disk or web is also analyzed. Discussions on the usefulness and limitations of the fieldmeter are presented.

I INTRODUCTION

When a large roll of film, called a jumbo, is unwound and moved through a manufacturing process, the film being transported in the process is referred to as a web. The manufacturing process modifies the web by one or more steps such as coating, curing, embossing, slitting, etc. [1], [2], [3]. During this processing the web comes in contact with cylinders, called rollers, which are usually grounded and some of which are used to move the web through the process as shown in Fig. 1.

At any given moment in time portions of the web are in contact with the rollers and the remaining portions of the web are between any contact with the rollers. When the separation between two adjacent rollers is not too small, the web in the central section between the rollers is referred to as a free-span of web [4].

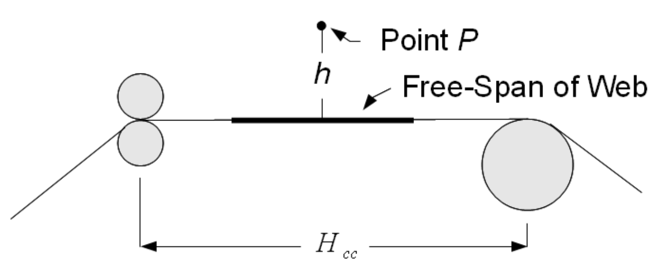


Figure 1: Free-Span of Web

When two materials come in contact and then separate, a charging of the surfaces can result; and this charging is called contact charging or triboelectrification [5]. A web – due to contact and separation from process rollers– will become charged; and, if the web is non-conducting, this charge will remain on the web for a long time [6]. As a result, a free-span of web can be considered a rectangular substrate (of dimensions L and W) that usually has some surface charges per unit area σ on it. If this surface charge density σ becomes high enough, an electrical spark can occur, and under certain conditions the spark can result in a shock, a fire or an explosion. Hence, for safety reasons it is required to keep σ low enough to prevent electrical sparks from occurring. According to Coulomb's Law (discussed in Section II) an electric field $\mathbf{E}$ is produced whenever charges are present and according to Gauss's Law when a second object approaches a charged surface there is ideally a simple Gauss's Law relationship (see (13) in Section V) between the magnitude of the field and the charges on the surface. Using (13) shows the electrical breakdown field (3 MV/m) occurs at $\sigma \approx 25$ μC/m^2, the hissing field (2 MV/m) occurs at $\sigma \approx 13$ μC/m^2, the uppermost safe field (1 MV/m = 10 kV/cm = 25.4 kV/inch) occurs at $\sigma \approx 8$ μC/m^2, and the "rule-of-safety" field (0.2 MV/m) for a processing line requires $\sigma \leq 1.7$ μC/m^2 [4, (p. 240)]. Consequently, in a process line it is important to determine σ as well as manage its control. (A note in passing: 1.6 μC/m^2= 10 charges/(μm)^2 so at spark-over or breakdown there are about 150 charges/(μm)^2 on the disk or web.)

Another situation exists in the semiconductor industry where a circular "300 mm wafer" (its diameter $D = 2R$ is actually 12 inches or 305 mm) of silicon is used as the starting substrate onto which is deposited material to eventually produced very large scale integrated (VLSI) circuits. During processing the wafer disk can become charged in several of the processing steps and produce damage in some of these circuits [7, (pp. 63-64)]. Hence it is prudent to keep σ low enough to prevent damage from occurring. Therefore, the magnitude of σ should be measured as a first step in detecting the existence of charge problems.

Still a third situation exists in fiber-making processes. Fibers have a large aspect ratio which is defined as their length to width ratio or L/W ratio. Defining $L/W = f$ allows the aspect ratio to be written as either $L : W$ or as $f : 1$ (two numbers separated by a colon).

When examining σ just a square in the center section of a free-span of web might be looked at in which case $f = 1$ and would be depicted as in Fig. 2a. Then, again, the free-span might be a wide web between not so large a separation of two rollers (top drawing of Fig. 2b), or, alternatively, a narrower web between a much larger separation of two rollers (bottom drawing of Fig. 2b). In both these situations the aspect ratio would be greater than unity. Finally the web in the free-span might be so thin that its aspect ratio becomes very large and it looses almost all sense of being called a web and is called a filament or fiber as depicted in Fig. 2c. It should be noted in passing that a disk as depicted in Fig. 2d will just fit inside a unit square as depicted in Fig. 2e but cover just slightly less area than the square. On the other hand, a square can just fit inside a disk as depicted in Fig. 2e but cover just slightly less area than the disk. This information will prove to be extremely useful and will be discussed further in Section V.

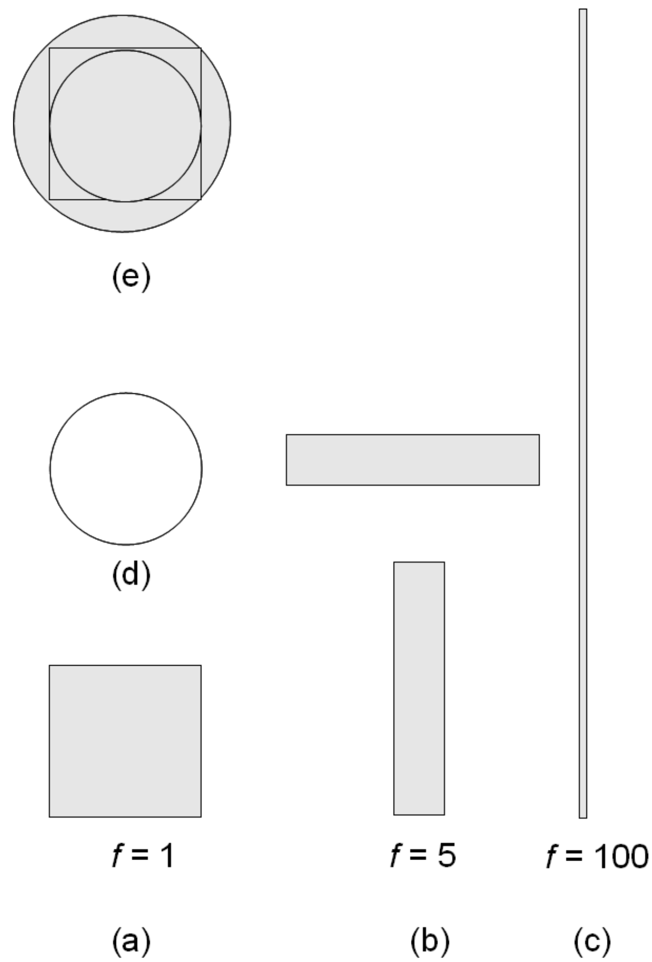


Figure 2: Aspect ratio for (a) square, (b) free-span web, and (c) filament

Coulomb's Law (discussed further in Section II) indicates an electric field $\mathbf{E}$ (discussed further in Section III) is produced whenever charges are present. Therefore, a charged disk or a charged free-span of web will produce an electric field. The normal component of the electric field $\mathbf{E}_n$ can be measured by an electrostatic fieldmeter [8]. A fieldmeter can be placed above an isolated disk or a free-span of web and the E-field reading obtained by the fieldmeter can be related back to the surface charge density σ. Gauss's Law, under the assumption of an infinitely-large and uniformly-charged disk or web, predicts the E-field is directly proportional to σ and non-varying with the height above the disk or web [9, (pp. 73-74)]. In other words, the reading of the fieldmeter is independent of the fieldmeter's position above an infinitely large disk or free-span of web (see (13) in Section V). In practice disks and free-spans of web are not infinitely large, and this positional independence of the fieldmeter is not found. Consequently, there exists a need to obtain a better understanding of the fieldmeter reading and its relationship to the surface charge density in real world situations. The goal of this paper is to calculate the electric field that exists at any height h above the center of an isolated but uniformly-charged *finite-sized* substrate (either a disk or a rectangular free-span of web as depicted in Fig. 1 and Fig. 4) and relate it to surface charge density σ on that substrate.

In order to obtain an equation relating the electric field to surface charge density the E-field will be determined at a point P above the center-point P' on the disk or free-span of web as depicted in Fig. 4. To do this the incremental normal component of the electric field $d\mathbf{E}_n$ at the point P will be determined due to an incremental charge $dQ = \sigma dS$ located on an incremental surface dS of the disk or web. Then the full normal electric field $\mathbf{E}_n$ will be determined by summing the incremental charge dQ over the whole surface of the disk or free-span of web. Finally, a fieldmeter will be placed at the point P and the expected E-field at the fieldmeter will be determined.

Although the total charge on a disk or web is often not completely uniform, the fieldmeter reading is essentially responding to an average of all the charges. Therefore, in this paper the charge on the disk or free-span of web will be assumed uniform and equal to this averaged σ. Also, in Fig. 1 if the web is moving from right to left, the bottom surface will become charged due to contact with the roller on the right. If instead the web is moving from left to right, both sides of the web become charged due to contact with the two rollers on the left. The fieldmeter is essentially measuring the E-field averaged from the charges on both surfaces, but the surface charge density from each surface can still be determined with an additional measurement [10].

II COULOMB'S LAW

A quick review of Coulomb's Law is useful in order to understand the concept of charges on disks and webs and how, as discussed in Section III, these charges produce an electric field. Coulomb showed by experiments that like charges repel and unlike charges attract and when two electrically charged small bodies q and Q are separated by a distance s, the force $\mathbf{F}$ of attraction or repulsion falls off as the distance squared and can be represented by [11, (p. 74)]

$$\mathbf{F} = F\hat{\mathbf{s}} = \frac{qQ}{4\pi\varepsilon s^2}\hat{\mathbf{s}}. \tag{1}$$

Here the direction $\hat{\mathbf{s}}$ of the force is along the line of centers of the two charges and ε is the permittivity of the material medium between the charges. When a charge q is at the location of q in Fig. 3A the vector $\hat{\mathbf{s}}$ points away from Q (upward in Fig. 3A) and when a charge Q is at the location of Q the vector $\hat{\mathbf{s}}$ points away from q (downward in Fig. 3A). The force is repulsive when the two charges q and Q have the same polarity; i.e., whenever both charges are positive or whenever both charges are negative. The force is attractive when the two charges q and Q have the opposite polarity; i.e., whenever the one charge is positive and the other is negative. Experiments also showed that when the two charges are separated by vacuum the force is stronger than if the charges are separated by a material. Each material placed between the charges has its own reducing effect. The permittivity of free-space (vacuum) is $\varepsilon_0 = 8.85 \text{ x } 10^{-12}$F/m and the permittivity of the material between the charges can be written as $\varepsilon = \varepsilon_0\varepsilon_r$ where the relative permittivity or dielectric constant ε_r is the ratio of two permittivities and is dimensionless. The greater the dielectric constant of the material between the charges the less the coulombic force. It is the dielectric constant (rather than its permittivity) that is reported in the literature as one property that classifies a material's electrical behavior.

Note that the force in $\mathbf{F}$ in (1) is a vector (vectors are written bold face) and it has a scalar magnitude F (magnitudes are written in italics) and a vector direction $\hat{\mathbf{s}}$ that conforms to the drawing in Fig. 3A. Since the magnitude F in (1) is proportional to the product of qQ, then, if q and Q are of opposite charge F will have a negative sign so $\mathbf{F}$ will point in the opposite direction to $\hat{\mathbf{s}}$. The vector $\hat{\mathbf{s}}$ (or any vector with a cap ˆ on it) is called a unit vector because its magnitude is unity (meaning it has a value of 1) and it does not change the magnitude of a scalar when, for example, the scalar F is multiplied by $\hat{\mathbf{s}}$ as in (1).

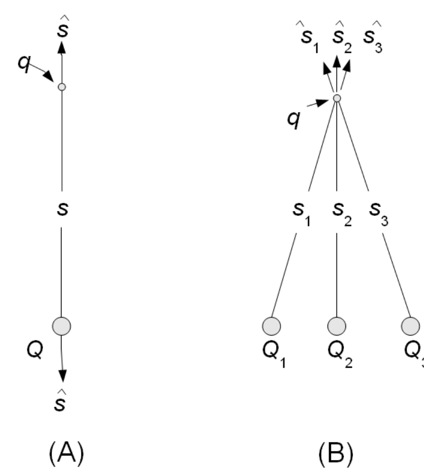
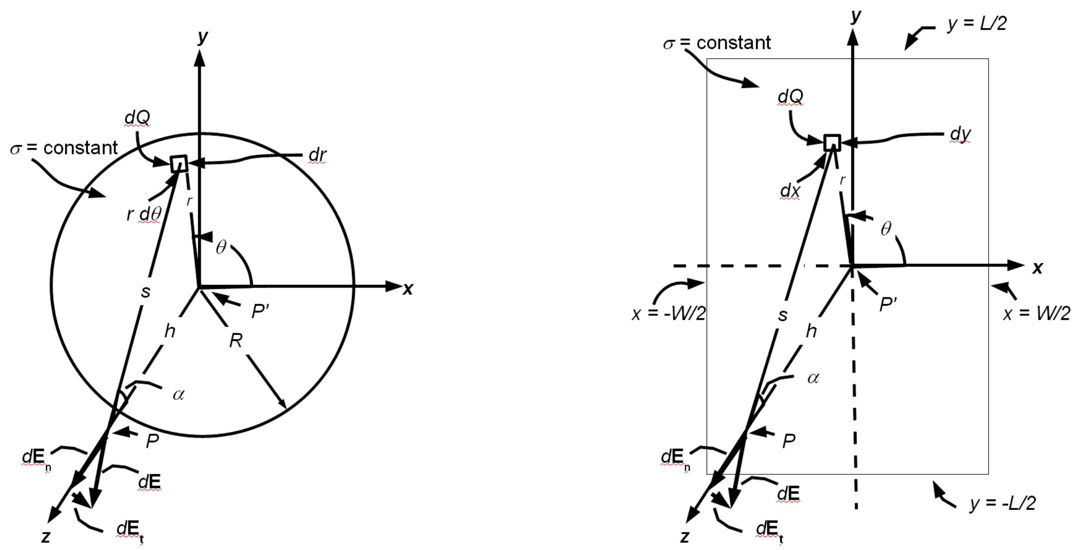


Figure 3: Diagrams depicting the vector force $\mathbf{F}$ in Coulomb's Law which is given by (1). In (A) there is a small charge Q located near q. In (B) there are several small charges the i^{th} having charge Q_i located near q. In Fig. 4a there is a small or incremental charge dQ on each small area dS of the disk where $dS = rd\theta dr$. In Fig. 4b there is a small or incremental charge dQ on each small area dS of the web where $dS = dydx$.



(a) Field at point P above the center of a uniformly charged disk.

(b) Field at point P above the center of a uniformly charged free-span of web.

Figure 4: Incremental Electric Field at a Height h at the Center Point P Due to an Incremental Charge dQ on (a) an Isolated and Uniformly Charged Non-Conducting Disk of Radius R or (b) an Isolated and Uniformly Charged Non-Conducting Free-Span of Web of Width W and Length L.

III ELECTRIC FIELD

Coulomb's Law (1) is useful to define the force between two charges located at two different points in space as depicted in Fig. 3A. However, most problems associated with electrostatics revolve around many charges and how they affect a charge located at a specific point in space as depicted in Fig. 3B. For example, suppose there is a charge q located at a point P in space. What is the force on q due to other charges Q_1, Q_2, Q_3,... located at different distances s_1, s_2, s_3,... (and with different directions $\hat{\mathbf{s}}_1$, $\hat{\mathbf{s}}_2$, $\hat{\mathbf{s}}_3$,...) from charge q? Also, what is the electric field at the location of q?

To answer these questions the charge q is moved to a point P where the force and E-field on q will both be evaluated. Assume that q has already been moved to point P in Fig. 3A; so, the coulomb force at the point P can then be calculated with (1). Next this force on q determined at the point P is divided by the charge q and the result is defined as the electric field (electric field force or E-field) at the point P due to the charge Q. Thus the electric field can be defined by

$$\mathbf{E} = \mathbf{F}/q. \tag{2}$$

If there are charges at many different locations as in Fig. 3B, then there is an electric field $\mathbf{E}_1 = \mathbf{F}_1/q$ due to charge Q_1 and a field $\mathbf{E}_2 = \mathbf{F}_2/q$ due to a charge Q_2 etc. and the total electric field at the point P where q resides is

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \ldots = \frac{1}{q}\left(\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots\right) \tag{3}$$

and the change in the field due to an addition of another charge (say the n^{th} charge) is

$$\triangle\mathbf{E} = \sum_{i=1}^{n}\mathbf{E}_i - \sum_{i=1}^{n-1}\mathbf{E}_i = \frac{1}{q}\left(\sum_{i=1}^{n}\mathbf{F}_i - \sum_{i=1}^{n-1}\mathbf{F}_i\right) = \frac{\mathbf{F}_n}{q}. \tag{4}$$

Thus, the electric field at a point P is (3) and is nothing more than the sum of all the coulomb forces at point P divided by the charge q at point P. Furthermore, the smallest or incremental change to the electric field (4) is simply due to the addition (or subtraction) of a single charge. The charge q located at the general point P in space comes into play so much in electrostatics that is referred to as a "test" charge [9, (p. 59)])

The next question to ask is: What is the electric field at a point P above the center of a free-span of web (or isolated disk) in a process line? When asking this question, it is customary to imagine placing a small positive test charge q at the point P and then finding the total electric field $\mathbf{E}$ exerted on this test charge by all the other charges that are around the process line. A free-span of web was chosen so that other charges (on equipment, etc.) will be far enough away so as not to significantly contribute to the field at point P. In other words, only the charges on the web contribute to the field at the point P which is located above the center of the free-span of web (or isolated disk). The force on the test charge from each of these web (or disk) charges is determined by Coulomb's law (1) and its E-field at point P is simply determined by (2). So, the total E-field is just the sum of the E-fields from each of the charges located in the region around the process line and hopefully that means just the charges on the free-span of web (or isolated disk).

IV FIELD FROM CHARGES

As stated in the Section I the goal is to measure the E-field above a charged surface and determine from the measurement the amount of surface charge density σ on the surface. The first step is to determine what the E-field should be at the measurement point P for any value of σ.

To determine what the field should be consider a uniformly-charged disk or a rectangular section of a uniformly-charged free-span of web as shown in Fig. 4. Suppose the electric field at a point P above the disk or web is desired. To obtain this field at point P first look at the electric field contributed from the incremental charge $\Delta Q \to dQ$ located on incremental area $\Delta S \to dS = rd\theta dr$ on the disk surface or $\Delta S \to dS = dydx$ on the web surface where either area is located a distance r from the center of the disk or the free-span of web as depicted in Fig. 4. The amount of incremental charge dQ is σdS where σ is the surface charge density defined by

$$\sigma = \lim_{\triangle s \to 0} \frac{\triangle Q}{\triangle S} = \frac{dQ}{dS}. \tag{5}$$

Referring to Fig. 4 and applying Coulomb's Law [9, (p. 59 and p. 71)] to the incremental addition of the charge dQ from incremental area dS on the disk or on the web gives the actual field increase dE, its normal component dE_n and its tangential component dE_t as

$$d\mathbf{E} = \frac{\sigma dS}{4\pi\varepsilon s^2}\hat{\mathbf{s}} = dE\hat{\mathbf{s}} \tag{6}$$

$$d\mathbf{E}_n = dE\cos\alpha\hat{\mathbf{n}} = dE\frac{h}{s}\hat{\mathbf{n}} = \frac{h\sigma dS}{4\pi\varepsilon s^3}\hat{\mathbf{n}} = dE_n\hat{\mathbf{n}} \tag{7}$$

$$d\mathbf{E}_t = dE\sin\alpha\hat{\mathbf{t}} = dE\frac{r}{s}\hat{\mathbf{t}} = \frac{r\sigma dS}{4\pi\varepsilon s^3}\hat{\mathbf{t}} = dE_t\hat{\mathbf{t}} \tag{8}$$

where (see Fig. 4a)

$$s^2 = h^2 + r^2 \tag{9}$$

in the cylindrical coordinate system used to analyze a disk or (see Fig. 4b)

$$s^2 = h^2 + x^2 + y^2 \tag{10}$$

for the rectangular coordinate system used to analyze a rectangular web.

Because the point P where the field is to be computed is located at the center of the disk or web, it is possible to make use of symmetry around the disk or web. For each incremental area dS to the left of center on the disk or web there will be a corresponding area dS located 180° and to the right. As a result, the tangential field (8) from each of these two areas will be equal in magnitude but opposite in direction and will cancel each other. Thus, the total tangential field will be zero at any distance h above the center point P'. Stated another way, at any distance h above the center point P' only the incremental normal field (7) will not "sum out", and the total field will be the sum of each incremental normal field (7).

V DISK DISCUSSION

Based on the information presented in Section IV the E-field at a point P located a distance h above the center of a disk of radius R (as shown in Fig. (1) and Fig. (4)a) is derived in Appendix A and found to be (23) and is transferred here as

$$E_{nD} = \left[\frac{\sigma}{2\varepsilon}\right]\left(1 - \frac{1}{\sqrt{1 + \left(\frac{D}{2h}\right)^2}}\right) \tag{11}$$

where $D = 2R$ is the diameter of the disk and h is the distance above the center of the disk to the point P where the specific value of the E-field is defined. As mentioned in Appendix A the subscript D is added to (11) as a reminder that (11) is only applicable in predicting the central field from an isolated and uniformly-charged substrate that is in the form of a *disk*.

A fieldmeter placed at a distance h above the disk will alter the field at point P as it brings the instrument's ground to point P above the disk. The disk will no longer be isolated. Since opposite charges attract, the conductive ground of the fieldmeter will result in counter-charges being brought to the fieldmeter's ground and sensing plate, these counter-charges having been attracted there by the charges on the disk. These counter-charges will also give rise to another electric field that will be detected by the fieldmeter. This second E-field will also be described by an equation of the form given by (11) but with $D \to D_g$ (i.e. the fieldmeter's sensing plate which acts as a ground plate) and $h \to h_g$ where h_g is the height from the sensing plate. In actual operation the sensing plate is exposed to the field and then exposed to a ground (no field) resulting in charges moving, through a current sensing circuit, to and from the sensing plate [8, (pp. 228-234)]. So, at the fieldmeter the sensing plate is detecting the field at $h_g = 0$. With $h_g = 0$ then (11) shows the field due to the fieldmeter's counter-charges gives a contribution $\sigma/(2\varepsilon)$ to the total E-field at the sensing plate. Since the counter-charges at the fieldmeter's sensing plate are of opposite polarity to those on the disk, the field from the sensing-plate's counter-charges will be in the same direction as the field from the disk. As a result, the field at the fieldmeter will be the sum of both electric fields and is given by

$$E_{mD} = \left[\frac{\sigma}{\varepsilon}\right]\left(1 - \frac{1}{\sqrt{1+\left(\frac{D}{2h}\right)^2}}\right) \qquad \text{E-field: Disk Center} \qquad (12)$$

where the subscript m on E_{mD} is a reminder that E_{mD} is the actual field that will be measured at the sensing electrode by the fieldmeter and includes the field contributed by the counter charges at the sensing electrode. Again, the subscript D is a reminder that (12) is only valid at the center of a *disk*.

The Gauss's Law solution for the electric field both above and below an isolated and uniformly-charged infinite-plane is $E_{G_i} = \left[\frac{\sigma}{2\varepsilon}\right]$ [9, (pp. 73-74)]. The subscript G on E_{G_i} is used here to indicate it is the E-field from a Gaussian infinite surface and the subscript i on the subscript G indicates it is an isolated surface (nothing else around).

On the other hand, the Gauss's Law solution for the electric field above a uniformly-charged infinite-disk which is located at any height h below an infinite ground plane is

$$E_G = \left[\frac{\sigma}{\varepsilon}\right] \qquad \text{E-field: Gauss Infinite Plane} \qquad (13)$$

and (13) is the Gaussian solution field that is usually assumed to be valid when making a fieldmeter measurement [4, (p. 236, Eq. 3)]. In other words, it is assumed that the disk is large enough to act as being infinite in extent and the fieldmeter acts as the infinite ground plane. Also notice (13) is independent of the height h where the fieldmeter is located.

However, (12) shows that only when $D \gg h$ will the fieldmeter give a reading consistent with the infinite-disk solution (13). For a *finite-sized* disk the fieldmeter will give a reading *less than* (13) because the fieldmeter cannot add up any field contributions from charges located beyond the disk. For a finite-sized disk, if (13) is assumed valid, a low reading on the fieldmeter might (incorrectly) imply that the amount of surface charge density σ is lower than the actual value. This is discussed in detail in the following disk example.

As a *disk example* consider a fieldmeter calibrated and held 2.54 cm (1 inch) above a 300 mm semiconductor industry wafer (actually a 30.5 cm or 12 inch diameter disk). The E-field at the center of the disk using (12) is $E_{mD} = \left(1 - 1/\sqrt{1+[30.5/\{(2)(2.54)\}]^2}\right)\left[\frac{\sigma}{\varepsilon}\right] = 0.84\left[\frac{\sigma}{\varepsilon}\right]$. This is a measured field that is only 84% of the field that would be measured from an infinite plane which is given by (13). If written as $E_{mD} = \left[\frac{0.84\sigma}{\varepsilon}\right]$ it can be seen that, if (incorrectly) (13) is assumed valid, the assumption would underestimate by 16% the actual surface charge density.

Most handheld fieldmeters today are calibrated to be held at 2.54 cm (1 inch) from the disk compared to 1 cm for older (or today's special order) fieldmeters. However, if a person were to use a fieldmeter calibrated to be held at $h = 1$ cm from the disk, then based on (12) the fieldmeter measurement would be $E_{mD} = \left(1 - 1/\sqrt{1+[30.5/\{(2)(1)\}]^2}\right)\left[\frac{\sigma}{\varepsilon}\right] = 0.93\left[\frac{\sigma}{\varepsilon}\right]$. Thus in the semiconductor industry, when making fieldmeter measurements on 300 mm wafers, the older or special order (1 cm calibration) fieldmeter gives less error ($\sim$7%) when assuming (13) is valid. Of course, what works best is not to assume (13) is valid, and instead use (12) for the specific disk diameter D being investigated with a fieldmeter (that has a specific calibration height h). In all studies it should also be kept in mind that the accuracy of a better fieldmeter is only about $\pm$3%. Finally, what is the surface charge density on the wafer if the older fieldmeter measures 1 kV/cm? Calculating $E_{mD} = 0.93\left[\frac{\sigma}{\varepsilon}\right]$ as before gives $\sigma = \varepsilon E_{mD}/0.93 = \left(8.85x10^{-12}\frac{F}{m}\right)\left(10^3\frac{V}{cm}\right)\left(10^2\frac{cm}{m}\right)\left(10^6\frac{\mu C}{C}\right)/0.93 = 0.95\mu C/m^2$.

The % error in the fieldmeter reading from the ideal Gaussian infinite-plane field (13) can be defined as

$$\% \text{ Disk Error} = \frac{E_G - E_{mD}}{E_G} \text{ x } 100 = \left(1 - \frac{E_{mD}}{E_G}\right) \text{ x } 100$$

where

$$\frac{E_{mD}}{E_G} = 1 - \frac{1}{\sqrt{1 + \left(\frac{D}{2h}\right)^2}} \qquad \text{Normalized E-field: Disk} \tag{14}$$

is the Gaussian normalized E-field *ratio* which can be simply referred to as the normalized E-field (for a disk).

In Fig. 5 the bottom curve shows a plot of (14) which is the expected normalized E-field from a finite disk vs. the half-diameter to fieldmeter-placement-height ratio; the $D/(2h)$ ratio or normalized half-width ratio. This dimensionless graph can be used to quickly determine how close the measured field will be to the infinite disk Gaussian situation defined by (13). Using the previous discussion of a 300 mm diameter wafer and a fieldmeter held at the recommended 2.54 cm height h gives $D/(2h) = 30.5/[(2)(2.54)] = 6$. Locating 6 on the horizontal axis in Fig. 5 and following upward to the lower curve gives a normalized E-field value of around 83% to 84% in agreement with the previous discussion. Likewise, for a fieldmeter calibrated and held at 1 cm the normalized half-width ratio $D/(2h)$ is 15.3. Locating 15 on the horizontal axis in Fig. 5 and following upward to the lower curve gives a normalized E-field value of around 0.93 or 93%, meaning the fieldmeter measurement will give a value of 93% of (13) in agreement with the previous discussion.

VI WEB DISCUSSION

Based on the information presented in Section IV the E-field at a point P located a distance h above the center of a free-span of web of width W and length L (as shown in Fig. (1) and Fig. (4)b) is derived in Appendix B and found to be (29) and is repeated here for ease of viewing as

$$E_{nW} = \frac{\sigma}{2\varepsilon}\left(\frac{2}{\pi}\tan^{-1}\frac{f\left(\frac{W}{2h}\right)^2}{\sqrt{\left(\frac{W}{2h}\right)^2(f^2+1)+1}}\right) \tag{15}$$

where, as mentioned in Section (I), f is the aspect ratio $L : W$ or $f : 1$ and is defined by $L = fW$.

Similar to the disk discussion of Section V the subscript W is added to (15) as a reminder that (15) is only applicable in predicting the field from the center of an isolated and uniformly-charged substrate that is in the form of *a rectangular free-span of web* of length L and width W.

For a free-span web that is being examined as a perfect square $f = 1$, and (15) reduces to

$$E_{nW\square} = \frac{\sigma}{2\varepsilon}\left(\frac{2}{\pi}\tan^{-1}\frac{\left(\frac{W}{2h}\right)^2}{\sqrt{2\left(\frac{W}{2h}\right)^2+1}}\right) \tag{16}$$

where the square symbol □ on the subscript of (16) signifies (16) only predicts the field from the center of an isolated and uniformly-charged square web of length W. If the free-span is not a square, then the field is given by (15).

As discussed in Section V when a fieldmeter is brought to the point P to make a measurement, the fieldmeter brings with it counter-charges which create a second electric field of magnitude $\sigma/(2\epsilon)$ that must be added to the measurement. As a result, the fieldmeter will measure an E-field of

$$E_{mW} = \frac{\sigma}{\varepsilon}\left(\frac{2}{\pi}\tan^{-1}\frac{f\left(\frac{W}{2h}\right)^2}{\sqrt{\left(\frac{W}{2h}\right)^2(f^2+1)+1}}\right) \qquad \text{E-field: Web Center} \tag{17}$$

where the subscript m on E_{mW} is a reminder that (17) is the actual field that will be measured by the fieldmeter. Similar to the discussion in Section V for the disk, here the subscript W in (17) is a reminder that (17) is only valid at the center of a *rectangular free-span of web* of length L and width W and aspect ratio $f : 1$ where f is defined by $L = fW$.

The % error in the fieldmeter reading from an ideal Gaussian infinite-plane measurement (13) can be defined as

$$\% \text{ Web Error} = \frac{E_G - E_{mW}}{E_G} \text{ x } 100 = \left(1 - \frac{E_{mW}}{E_G}\right) \text{ x } 100$$

where

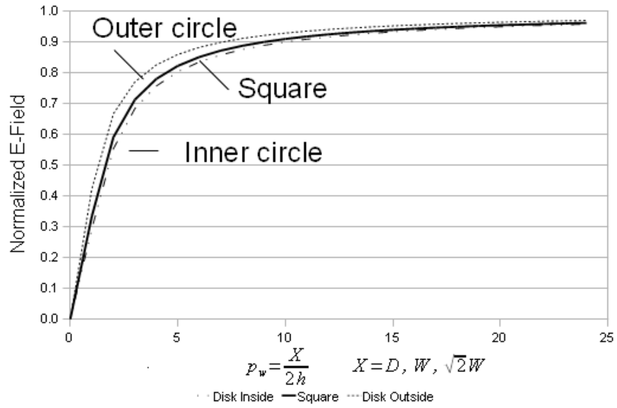


Figure 5: Comparison of the normalized E-field from the center of a square to those of disks located just inside and just outside the square as depicted in Fig. 2e. If the disk and square were of infinite extent, then their normalized fields would be 1.0 meaning 100% of (13). A fieldmeter held 2.54 cm (1 inch) above a 300 mm (12 inch) diameter disk D would have $D/(2h) = 6$ whereas a 300 mm square of web would also have $W/(2h) = 6$.

$$\frac{E_{mW}}{E_G} = \left(\frac{2}{\pi} \tan^{-1} \frac{f\left(\frac{W}{2h}\right)^2}{\sqrt{\left(\frac{W}{2h}\right)^2 (f^2+1)+1}} \right) \qquad \text{Normalized E-field: Web} \qquad (18)$$

is the Gaussian normalized E-field *ratio* which can be simply referred to as the normalized E-field (for a web).

Fig. 5 is a plot (using (18) with $f = 1$) of the normalized E-field expected from the fieldmeter for a square of uniformly-charged web (solid line), and this square is depicted in Fig. 2a and Fig. 2e. Also shown in Fig. 5 is a plot of (14) for a disk of diameter $D = W$ (lower curve, dot-dot-dash line), and this disk is depicted by the inner disk in Fig. 2e. Finally, in Fig. 5 is a plot of (14) for a disk of diameter $D = \sqrt{2}W$ (uppermost curve, dot-dot-dot line), and this disk is depicted by the outer disk in Fig. 2e. Thus, the three curves in Fig. 5 give the normalized fields from the inner disk, the square and the outer disk depicted in Fig. 2e. In Fig. 5 the inner disk (lower curve) being located inside the square (middle curve) has a lower normalized E-field than the square because this disk does not count the charges in the corners of the square as can be seen in Fig. 2e. Likewise, in Fig. 5 the square (middle curve) being located inside the larger disk (upper curve) has a lower normalized E-field than the larger disk because the square does not count the charges outside of the square.

Because the three curves in Fig. 5 are reasonably close at high normalized half-width ratios, it suggests that to a first approximation measuring the field in the center of a very large web within a free-span can be thought of as measuring the field from a very large disk located within the web of the free-span. If the mind's eye visualizes a disk that fits within the web, all that is necessary is to make sure the $D/(2h)$ ratio of this mind's eye disk is greater than (say) about 18. Then from Fig. 5 it can be seen that for $D/(2h) \geq 18$ the field measured from the charges just within this disk will have summed up enough of the field to be closer than about 95% of (13). In other words, even if the web were infinitely large all the charges outside this mind's eye disk would only contribute less than 5% more to the field. It is Coulomb's Law in action; closer charges have a greater force and greater E-field.

Most fieldmeters are calibrated for h at 2.54 cm (1 inch); so, $D/(2h) \geq 18$ or $D \geq 36h$ means the E-field collected from a mind's eye disk of $D \geq 36$ inches will give 95% of (13). So it can be concluded that, when using an $h = 2.54$ cm (1 inch) calibrated fieldmeter, if a 0.9 m (1 yd) diameter disk can be visualized in the mind's eye as fitting within the web of a free-span, then the measured E-field will better than 95% of the E-field described by (13). For this situation (13) can be used to calculate the surface charge density σ from the fieldmeter measurement. If using a fieldmeter calibrated at $h = 1$ cm, then $D \geq 36h$ gives a mind's eye disk of $D \geq 36$ cm (14 inch) diameter for 95% of (13).

From the preceding an electrical definition for a "Gaussian free-span" of a large width web can be developed. Let $D/(2h) \geq 18$ (or $D \geq 36h$) be chosen as the criterion above which little (≤5%) change in fieldmeter reading is expected. Then as depicted in Fig. 1 the center-to-center distance of two rollers of radii r_1 and r_2 should be no closer than $H_{cc} = r_1 + r_2 + C_{PL}D$ where D is the diameter of the mind's eye disk visualized to fit within the area of the web on the free-span. Here, $C_{ML} \geq 1$ is the manufacturing line constant which is unity if other surroundings near the fieldmeter have little or no charge on them. For example, if using a fieldmeter that is calibrated at 2.5 cm (1 inch), then the criterion $D \geq 36h$ gives $D \geq$0.9 m (36 inches or 1 yd). If the fieldmeter is calibrated at 1 cm, then $D \geq 36h$ gives $D \geq$36 cm (15 inches). In other words, for a fieldmeter that is calibrated to measure the field at a height h above the free-span of web, the free-span can be called a "Gaussian free-span" if $H_{cc} \geq r_1 + r_2 + 36C_{ML}h$ and an imaginary or mind's eye disk of diameter $D = 36h$ can be visualized to fit completely within

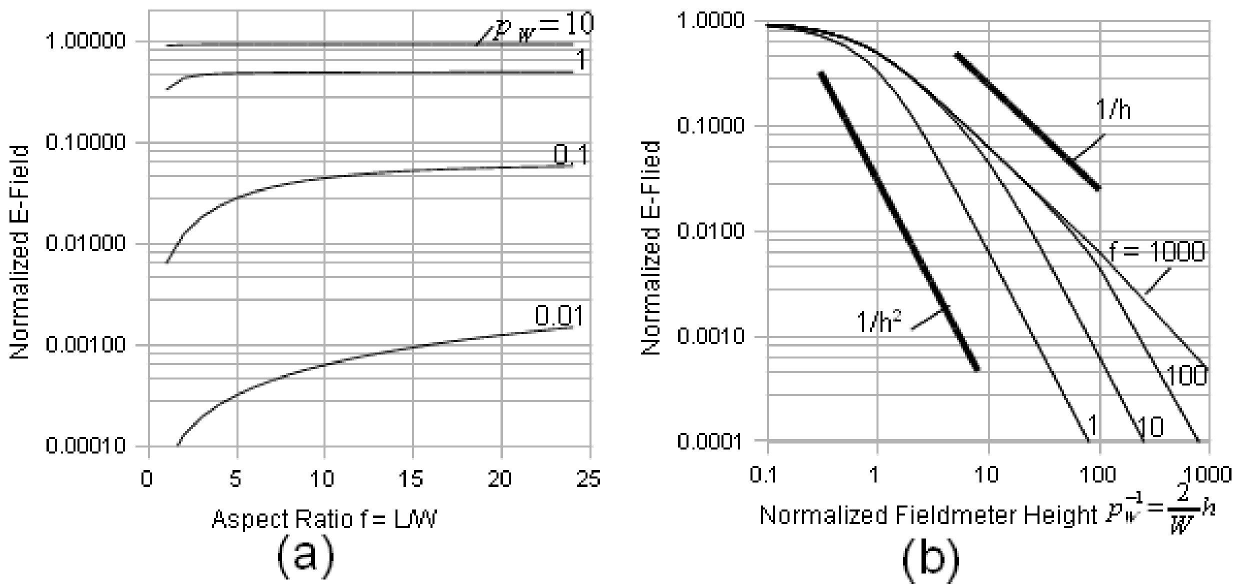


Figure 6: Aspect Ratio Graphs. Graph (a) shows a web with $p_w = W/(2h)$ for 4 decades of decreasing web width W (assuming h constant) or increasing distance h from the web (assuming W constant). Graph (b) has two heavy solid lines showing the E-field's classical charged wire fall off of $1/h$ and classical point charge fall off of $1/h^2$. Fibers of very long aspect ratio f 100 to 1000 behave like a charged wire and small aspect ratio fibers $f = 10$ fall off quicker and a fiber stub $f = 1$ behaves like a point charge.

the web's area on the free-span. For a "Gaussian free-span" (13) may be used to estimate the surface charge density σ from the fieldmeter reading and the result will be within 5% (worst case: the estimated σ will be approximately 5% *low*). The multiplying constant C_{ML} should also include the fieldmeter manufacturer's suggested distance that their fieldmeter should be kept from other charged objects.

If the mind's eye disk $D = 36h$ when visualized as placed just beneath the web is not *completely* covered by the web in the free-span, then although the web may be in a free-span it is not a "Gaussian free-span" and (17) must be used. These narrow webs and filaments require special attention and are discussed in greater detail in Section VII.

Finally, it has also been reported that the field from the center of an isolated square of a substrate having uniform charge should be [9, (p. 106, Problem 2.41)]

$$E_{n\square} = \left[\frac{\sigma}{2\varepsilon}\right]\left\{\frac{4}{\pi}\tan^{-1}\sqrt{\frac{1}{2}\left(\frac{W}{h}\right)^2 + 1} - 1\right\} \tag{19}$$

where again the square symbol $\square$ on the subscript of $E_{n\square}$ indicates this is the normal component of the electric field at the center of a square substrate. A spreadsheet analysis of (19) and (16) reveals that both equations predict the identical field as a function of $W/(2h)$ even though the equations look slightly different.

VII ELONGATED WEB DISCUSSION

A free-span of web as depicted in Fig. 1 is a section of web that can be classified by its aspect ratio $L : W$ or $f : 1$ where $L = fW$. The web section could be a wide web located between rollers spaced just far enough apart to define the web as a free-span as depicted in Fig. 2b (top figure: imagined with horizontal rollers located above and below the figure). The web section could also be a narrow web located between rollers spaced far enough apart to define the web as a free-span as depicted in Fig. 2b (bottom figure). Finally the web section could be a very narrow web located between rollers or other fixtures and spaced at least far enough apart to define the web as a free-span but also to define the web more as a filament or fiber as depicted in Fig. 2c. The electric field from all such webs will be given by (15) and the measured E-field will be given by (17). It is easiest to describe the fieldmeter behavior by the normalized E-field (18).

The normalized field as a function of web aspect ratio is shown in Fig 6a. Only the $p_w = W/(2h)$ equals 10 curve is in the range of 0.9 or 90% of (13). For all other smaller p_w ratios (i.e., half-width to fieldmeter placement ratios) the normalized field drops significantly indicating the use of (13) with the E-field measurement would be incorrect without a sizable correction factor.

Examples are the best way to describe the use of the information in Fig 6. Consider, as a *first example*, a web that is 50 cm (20 inch) wide and situated in a free-span of 2 m (80 inches). In the free span the web aspect ratio is $f = L/W = 200/50 = 4$. For a fieldmeter calibrated at 2.5 cm (1 inch) the normalized half-width ratio is $p_w = W/(2h) = 50/[(2) \cdot (2.5)] = 10$. From Fig. 6a the curve for $p_w = 10$ and an aspect ratio of 4 gives a normalized E-field correction factor of about 0.9. Likewise, in Fig. 6b an aspect ratio of $f = 4$ will be a curve that fits between $f = 1$ and $f = 10$. Since $p_w = 10$ then the normalized fieldmeter height ratio $p_w^{-1} = 1/10 = 0.1$. Thus in Fig. 6b the $f = 4$ curve meets $p_w^{-1} = 0.1$ at a normalized E-field correction factor of about 0.9 in agreement with Fig. 6a. Also note that in Fig. 6b, if the fieldmeter is moved away from the standard 2.5 cm spacing and moved to a height $h = 25$ cm, then $p_w^{-1} = 1$ and the slope of the $f = 4$ curve at $p_w^{-1} = 1$ falls off as $1/h$ indicating the free-span of web is now looking like a cylinder. As the fieldmeter moves another decade away to $h = 250$ cm, then $p_w^{-1} = 10$ and in Fig. 6b the slope of the $f = 4$ curve at $p_w^{-1} = 10$ falls off as $1/h^2$ indicating the free-span of web is now acting like a small spot (point charge). This gives a general feel of what the fieldmeter senses, namely a 200 cm x 50 cm web has an E-field that falls off like a point charge at 250 cm.

As a *second example*, assume the web is in the form of a 5 mm (0.2 inch) wide filamentary ribbon situated in a free-span of 1 m (40 inches). In the free span the ribbon's aspect ratio is $f = L/W = 100/0.5 = 200$. For a fieldmeter calibrated at 2.5 cm (1 inch) the normalized half-width ratio is $p_w = W/(2h) = 0.5/[(2) \cdot (2.5)] = 0.1$. From Fig. 6a the curve for $p_w = 0.1$ does not show an aspect ratio of 200 but at 25 it is clear the $p_w = 0.1$ curve is heading towards a normalized E-field correction factor of 0.06. Likewise, in Fig. 6b an aspect ratio of $f = 200$ will be a curve that fits between 100 and 1000 and these two curves fit almost on top of each other until $p_w^{-1} \approx 70$. Since $p_w = 0.1$ then $p_w^{-1} = 1/0.1 = 10$. Thus the $f = 200$ curve meets $p_w^{-1} = 10$ at a normalized E-field correction factor of about 0.06 in agreement with Fig. 6a. Also note that at $p_w^{-1} = 10$ the fieldmeter senses a web that looks like a charged wire with a field that is falling off as $1/h$. But if the fieldmeter moved outward to $h = 25$ cm, then $p_w^{-1} = 100$, and the slope of the $f = 200$ curve at $p_w^{-1} = 100$ is approaching a fall off of $1/h^2$ and the fieldmeter senses a (100 cm x 0.5 cm) free-span of web that is now starting to look more like a point charge. The most important question to ask is as follows: What is the surface charge density in this second example if at $h = 2.5$ cm (1 inch) the fieldmeter reading is 0.6 kV/in? (18) requires a spreadsheet or newer scientific calculator. Since Fig. 6 was created with (18), then $E_{mD} = (0.06)\, E_G = (0.06)\, \sigma/\varepsilon$ and $\sigma = \varepsilon E_{mD}/(0.06)$ so $\sigma = \left(8.85x10^{-12}F/m\right)\left(0.6x10^3V/inch\right)(inch/2.54cm)\left(10^2cm/m\right)\left(10^6\mu C/C\right)/(0.06) = 3.5\mu C/m^2$. From the information on surface charge presented in Section I, it can be concluded that this is a highly charged web even though the measured field seemed reasonably small.

As a *third example*, consider any fiber having a diameter below (say) 500 µm (20 mils). For a fieldmeter calibrated at 2.5 cm (1 inch) the half-width to fieldmeter placement ratio is $p_w = W/(2h) \leq 0.05/[(2) \cdot (2.5)] \leq 0.01$. From Fig. 6a any curve for $p_w \leq 0.01$ will be three orders of magnitude or more below a normalized E-field of unity. This is at or below the limit of sensitivity of even the best fieldmeters, so present day fieldmeters are not capable of measuring the charge on a single fiber.

VIII CONCLUSIONS

Equations have been developed for the electric field at the center of an isolated and uniformly-charged disk (11) and an isolated and uniformly-charged free-span (length L, width W) of web (15).

Equations have also been developed for the expected measured E-fields from the disk (12) and from the web (17).

It was shown in Section V that about 95% of the E-field at a height h above the center of a uniformly-charged flat-substrate is contributed by the charges within a central region inscribed by either a disk of diameter D or a square of side W and given by $W = D = 36h$.

The criteria for a working definition of a "Gaussian free-span" of web was developed in Section VI. The criteria states, when using a fieldmeter that is calibrated for a fieldmeter-to-web distance of height h, the center-to-center distance between rollers depicted in Fig. 1 must be at least $H_{cc} \geq r_1 + r_2 + 36C_{ML}h$ where r_1 and r_2 are the radii of the two rollers and $C_{ML} \geq 1$. With this criteria, if a mind's-eye circle of diameter $D = 36h$ can be drawn beneath the free-span of web; and, if the circle becomes completely covered by the charged web, then the measured field can be directly related to the surface charge density σ using (13) to within an accuracy of 5%; namely, σ will be underestimated by less than 5%.

If only the H_{cc} criterion for a "Gaussian free-span" of web is met, but the mind's eye circle is not completely covered by the web, then (17) must be used. Full-blown examples of the use of (17) are given in Section VII including the calculation of σ from the measured E-field.

For a free-span of a high aspect ratio material such as thin web in a free-span the dimensionless curves of Fig. 6, which are based on (17), can be used to obtain an estimate of σ from the measured E-field. Unfortunately, it is shown in Section VII that the sensitivity of present-day fieldmeters is too low to detect an E-field from any web with a width below 500 µm, but the information in Section VII should prove useful for these fibrous materials once a more sensitive fieldmeter is developed.

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A DISK APPENDIX

The addition of an incremental charge $dQ = \sigma dS = \sigma r d\theta dr$ on the surface of the disk gives an incremental change to the normal component of the electric field (7). By combining the fact that $s^3 = \left(s^2\right)^{3/2}$ with (9) allows (7) to be written as

$$dE_{nD} = \left[\frac{h\sigma}{4\pi\varepsilon}\right] \frac{r d\theta dr}{\left(h^2 + r^2\right)^{3/2}} \tag{20}$$

where the subscript D on (20) is placed as a reminder that this is the incremental field from a disk.

The total normal component of the electric field at point P is the sum due to all incremental charges on the disk surface which is $E_{nD} = \int dE_{nD}$ or

$$E_{nD} = \left[\frac{h\sigma}{4\pi\varepsilon}\right] \int_0^{2\pi} \left\{ \int_0^R \frac{r dr}{\left(h^2 + r^2\right)^{3/2}} \right\} d\theta \tag{21}$$

where the terms inside the square brackets are constant. Defining $X = a + br + cr^2$ with $a = h^2$, $b = 0$, $c = 1$, and $q = 4ac - b^2$ the solution to the squiggly bracketed $\{\ \}$ integral in (21) can be found [12, (pp. 23-24, 170)], [13, (p. A-46, 247)] as

$$\int \frac{r dr}{X\sqrt{X}} = -\frac{2\left(br + 2a\right)}{q\sqrt{X}}$$

so integrating (21) with respect to r gives

$$E_{nD} = \left[\frac{h\sigma}{4\pi\varepsilon}\right] \int_0^{2\pi} \left\{ \frac{1}{h} - \frac{1}{\sqrt{h^2 + R^2}} \right\} d\theta$$

and integrating with respect to θ gives

$$E_{nD} = \left[\frac{\sigma}{2\varepsilon}\right] \left(1 - \frac{1}{\sqrt{1 + \frac{R^2}{h^2}}}\right) \tag{22}$$

$$E_{nD} = \left[\frac{\sigma}{2\varepsilon}\right] \left(1 - \frac{1}{\sqrt{1 + \frac{R^2}{h^2}}}\right). \tag{23}$$

Note that when $R \to \infty$ (23) goes to $E_{nD} = \left[\frac{\sigma}{2\varepsilon}\right]$. This is in agreement with the Gauss's Law solution for the electric field both above and below an isolated infinite plane carrying a uniform surface charge [9, (pp. 73-74)].

Recall the series expansion [12, (p. 88, 748)] $(1 \pm x)^{-n} = 1 \mp nx + \frac{n(n+1)}{2!}x^2 \mp \frac{n(n+1)(n+2)}{3!}x^3 + \cdots (\mp)^k \frac{(n+k-1)!}{(n-1)!k!}x^k + \cdots$ for $x^2 < 1$. Thus, when $x \ll 1$ and $n = \frac{1}{2}$ the expansion gives $(1 + x)^{-1/2} \approx 1 - \frac{1}{2}x$.

Hence, when $R \to\approx 0$, $\left(1 + \frac{R^2}{h^2}\right)^{-1/2} \approx 1 - \frac{1}{2}\frac{R^2}{h^2}$ and (23) goes to

$$E_{nD} \approx \left[\frac{\sigma}{2\varepsilon}\right] \left(1 - \left[1 - \frac{1}{2}\frac{R^2}{h^2}\right]\right) \approx \left[\frac{\sigma}{2\varepsilon}\right] \left(\frac{1}{2}\frac{R^2}{h^2}\right) \approx \left[\frac{\sigma}{4\pi\varepsilon}\right] \left(\frac{\pi R^2}{h^2}\right) \approx \frac{Q}{4\pi\varepsilon h^2}$$

which is the coulomb field of a point charge Q that has the characteristic fall off of the reciprocal of the square of distance (in the present case $1/h^2$) from the point charge [9, (p. 65)].

B WEB APPENDIX

The addition of an incremental charge $dQ = \sigma dS = \sigma dy dx$ on the surface of a free span of web gives the incremental change to the normal component of the electric field as (7). By combining the fact that $s^3 = \left(s^2\right)^{3/2}$ with (10) allows (7) to be written as

$$dE_{nW} = \left[\frac{h\sigma}{4\pi\varepsilon}\right] \frac{dy dx}{\left(h^2 + x^2 + y^2\right)^{3/2}} \tag{24}$$

where the subscript W on (24) is placed as a reminder that this is the incremental field from a web.

The total normal component of the electric field at point P is the sum due to all incremental charges on the web surface which is $E_{nW} = \int dE_{nW}$ or

$$E_{nW} = \left[\frac{h\sigma}{4\pi\varepsilon}\right] \int_{-L/2}^{L/2} \left\{ \int_{-W/2}^{W/2} \frac{1}{\left(h^2+x^2+y^2\right)^{3/2}} dx \right\} dy. \tag{25}$$

Multiplying the integral inside the squiggly brackets { } by $h^3/h^3 = \left(h^2\right)^{3/2}/h^3 = \left(\frac{1}{h^3}\right)/\left[\frac{1}{\left(h^2\right)^{3/2}}\right]$ and then letting $u = x/h$, $v = y/h$ and $p_W = W/(2h)$ and $p_L = L/(2h)$ gives

$$E_{nW} = \left[\frac{\sigma}{4\pi\varepsilon}\right] \int_{-p_L}^{p_L} \left\{ \int_{-p_W}^{p_W} \frac{1}{\left(\left[1+(v)^2\right]+(u)^2\right)^{3/2}} du \right\} dv. \tag{26}$$

The integration of the du term inside the squiggly brackets { } of (26) is accomplished by noting v is temporarily a constant during the du integration. Therefore, the square brackets term $\left[1+(v)^2\right]$ in (26) is temporarily a constant. Referring to (26) define $X = a + bu + cu^2$ and let $a = 1 + (v)^2$, $b = 0$ and $c = 1$ and $q = 4ac - b^2$. Then [12, (p. 23-24, 162)], [13, (p. A-45, 239)] give the solution $\int \frac{1}{X\sqrt{X}} du = \frac{2(2cu+b)}{q\sqrt{X}}$ which can be applied to (26) to yield

$$E_{nW} = \frac{\sigma}{4\pi\varepsilon} \int_{-p_L}^{p_L} \left(\frac{2p_W}{\left[1+(v)^2\right]\sqrt{\left[(p_W)^2+1\right]+(v)^2}} \right) dv. \tag{27}$$

To integrate over dv it is helpful to note that [12, (p. 33, 299)] gives the following integral solution

$$\int \frac{dv}{\left\{a'+c'(v)^2\right\}\sqrt{a+c(v)^2}} = \frac{1}{a'}\sqrt{\frac{a'}{ac'-a'c}} \tan^{-1} v \sqrt{\frac{ac'-a'c}{a'\left(a+cv^2\right)}}.$$

So to get (27) into this solution form let $a' = 1$, $c' = 1$, $a = p_W^2 + 1$ and $c = 1$ and then integrating with respect to v gives

$$E_{nW} = \frac{\sigma}{2\varepsilon}\left(\frac{2}{\pi}\tan^{-1}\frac{p_L p_W}{\sqrt{\left(p_W^2+p_L^2+1\right)}}\right) \tag{28}$$

where the identity [14, (p. 80, 4.4.16)] $\tan^{-1}(-z) = -\tan^{-1}(z)$ was used after the integration.

In Section (V) the field could be defined by a single ratio of disk radius R to height h of the point P where the field will be measured. In terms of the disk diameter D this ratio is $R/h = D/(2h)$. To describe a rectangular web in a similar ratio let $L = fW$, then, since $p_W = W/(2h)$, so must $p_L = L/(2h) = fW/(2h) = fp_W$ and (28) becomes

$$E_{nW} = \frac{\sigma}{2\varepsilon}\left(\frac{2}{\pi}\tan^{-1}\frac{fp_W^2}{\sqrt{p_W^2\left(f^2+1\right)+1}}\right) = \frac{\sigma}{2\varepsilon}\left(\frac{2}{\pi}\tan^{-1}\frac{f\left(\frac{W}{2h}\right)^2}{\sqrt{\left(\frac{W}{2h}\right)^2\left(f^2+1\right)+1}}\right). \tag{29}$$

Note in (29) that if $W \to \infty$ then since $p_W = W/(2h)$ $p_W \to \infty$ and $\frac{fp_W^2}{\sqrt{p_W^2(f^2+1)+1}} \to p_W \to \infty$ as long as f remains finite. Therefore, when $W \to \infty$ the $\tan^{-1}$ term in (29) becomes $\tan^{-1}\infty = \frac{\pi}{2}$ and (29) reduces to

$$E_{nW} = \frac{\sigma}{2\varepsilon}\left[\frac{2}{\pi}\tan^{-1}\infty\right] = \frac{\sigma}{2\varepsilon}\left[\frac{2}{\pi}\left(\frac{\pi}{2}\right)\right] = \frac{\sigma}{2\varepsilon}$$

which is independent of h and is the Gauss's law solution for an infinite plane carrying a uniform surface charge σ.

At the other extreme note that if $W \to\approx 0$ then since $W \ll 1$ so to $p_W = \frac{W}{2h} \ll 1$ and $\frac{fp_W^2}{\sqrt{p_W^2(f^2+1)+1}} \to fp_W^2$ and $fp_W^2 \ll 1$ but the series [14, (p. 88, 4.6.33)] for $|x| < 1$, is $\tan x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots$ so for $W \to\approx 0$, $\tan^{-1}\frac{fp_W^2}{\sqrt{p_W^2(f^2+1)+1}} \approx fp_W^2$ and

$$E_{nW} \to \frac{\sigma}{2\varepsilon}\frac{2}{\pi}\cdot fp_W^2 = \frac{\sigma}{\pi\varepsilon}\cdot f\left(\frac{W}{2h}\right)^2 = \frac{\sigma}{\pi\varepsilon}\cdot\frac{L}{W}\left(\frac{W}{2h}\right)^2 = \frac{\sigma LW}{4\pi\varepsilon h^2} = \frac{Q}{4\pi\varepsilon h^2}$$

which varies as $1/h^2$ and is in agreement with the coulomb field for a point charge Q [9, (p. 65)].